Math 259A Lecture Notes

Professor: Sorin Popa
Scribe: Daniel Raban

Contents

1 Introduction to Operator Algebras

1.1 $*$-algebras

Let H be a Hilbert space. We denote $B(H)$ to be the space of operators on H: $B(H)$ is the set of $T : H \to H$ such that $\sup_{\xi \in (H)_1} \|T(\xi)\| =: \|T\| < \infty$, where $(H)_1$ is the closed unit ball. $B(H)$ is an algebra.

Definition 1.1. An **operator algebra** is a vector subspace $B \subseteq B(H)$ closed under multiplication.

Given an operator T, we have an **adjoint operator** T^* which satisfies $\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle$ for all $\xi, \eta \in H$. The adjoint has $\|T^*\| = \|T\|$. This defines an operation $* : B(H) \to B(H)$ sending $T \mapsto T^*$. The $*$ operation satisfies

- $(T+S)^* = T^* + S^*$
- $(\lambda T)^* = \overline{\lambda} T^*$
- $(TS)^* = S^* T^*$
- $(T^*)^* = T$.

Definition 1.2. $B \subseteq B(H)$ is a **$*$-algebra** of operators on $B(H)$ if it is closed under the $*$ operation.

Example 1.1. Look at $B(\ell^2_\infty) = M_\infty(\mathbb{C})$. ℓ^2_∞ has an orthonormal basis e_i with $(e_i)_j = \delta_{i,j}$. Elements of $B(\ell^2_\infty)$ can be multiplied like infinite matrices, and the entries can be determined by this orthonormal basis.

We always consider algebras with a unit. So $B \subseteq B(X)$ will always contain the element $1_B = \mathrm{id}_X \in B$.

1.2 von-Neumann algebras and group von-Neumann algebras

Definition 1.3. A **von-Neumann algebra** is a $*$-algebra $B \subseteq B(X)$ closed in the weak operator topology given by the seminorms $p_{\xi,\eta}(T) = |\langle T\xi, \eta \rangle|$ ($T_i \to T$ in the weak operator topology if $\langle T_i(\xi), \eta \rangle \to \langle T(\xi), \eta \rangle$ for all $\xi, \eta \in X$).

Example 1.2. $B(X)$ is a von-Neumann algebra.

Definition 1.4. An operator $U \in B(X)$ is **unitary** if $U^* = U^{-1}$.

Example 1.3. A representation $\pi : \Gamma \to B(X)$ of a group Γ is called **unitary** if $\pi(g)$ is unitary for all $g \in \Gamma$. If π is unitary, then $\operatorname{span} \pi(\Gamma)$ is a $*$-algebra on X. Then the closure of this space under the weak operator topology is a von-Neumann algebra.

Denote $\ell^2(I)$ as ℓ^2 with an orthonormal basis indexed by I.

Example 1.4. Define the following representations of Γ

1. The **regular representation** is $\lambda : \Gamma \to U(\ell^2(\Gamma))$ is $\lambda(g)\xi_h = \xi_{gh}$
2. Alternatively, right group multiplication induces the unitary representation $\rho : \Gamma \to U(\ell^2(\Gamma))$ given by $\rho(g)\xi_h = \xi_{hg^{-1}}$.

Observe that $[\lambda(g_1), \rho(g_2)] = 0$. Let $L(\Gamma)$ be the weak operator topology closure of $\mathrm{span}(\lambda(\Gamma))$, and let $R(\Gamma)$ be the weak operator topology closure of $\mathrm{span}(\rho(\Gamma))$. These are **left and right group von-Neumann algebras**. One avenue of study to study the map $\Gamma \mapsto L(\Gamma)$.

This has many applications. These operators arising from groups are related to dynamics and ergodic theory.

1.3 Factors and C^*-algebras

Definition 1.5. A von-Neumann algebra M is a **factor** if $Z(M) = \mathbb{C}_1$, where Z denotes the center of the algebra.

Example 1.5. $B(X)$ and $L(\Gamma)$ are factors.

Here is a question that appeared early in the theory of von-Neumann algebras: Are there any other von-Neumann factors than $B(X)$?[1] This is fundamental to understanding how much commutation there is in operator algebras. The answer is yes. In fact, $L(\mathbb{F}_2)$ and $L(S_\infty)$ are not isomorphic to $B(X)$.

These two are infinite dimensional factors, and they have a trace functional on them, $\tau : M \to \mathbb{C}$ which is linear and continuous such that $\tau(x, y) = \tau(yx)$ for all $x, y \in M$. In general, if X is infinite dimensional, $B(X)$ has no trace defined everywhere.

Another question: Can we axiomatize the theory of von-Neumann algebras? We have a Banach-algebra with the $*$-operation, and we have the norm with $\|T^*\| = \|T\|$. Can we construct a Hilbert space only from this information?[2]

Definition 1.6. A $*$-algebra $B \subseteq B(X)$ of operators on X is called a **(concrete) C^*-algebra**.

In fact, these satisfy $\|T^*T\| = \|T\|^2$ for all T. (This does imply that $\|T^*\| = \|T\|$.)

[1] von-Neumann asked this question in 1935. He gave this question to a postdoc. Prior to this, he knew that any von-Neumann algebra decomposes via a measurable field of matrices as $M \cong \int_X M_t \, dt$. They solved the problem in 1936.

[2] Gelfand and Naimark worked on this in 1940-1943. They did not succeed, and Grothendieck tried in the 50s.

Definition 1.7. A Banach algebra with $*$ satisfying $\|x^*x\| = \|x\|^2$ is called an **abstract C^*-algebra**

Theorem 1.1 (G-N + Segal, 1943)**.** *If B is an abstract C^* algebra, then it is a concrete C^*-algebra.*

2 Introduction to C^*-Algebras

2.1 Recap

Recall: We are interested in the following objects.

Definition 2.1. A ***-algebra** M is an algebra with an involution $*$ (called the **adjoint**) such that if $T \in M$, then $T^* \in M$.

Definition 2.2. A **von Neumann algebra** $M \subseteq B(H)$ is a *-algebra of operators on a Hilbert space with $1 = \mathrm{id}_H \in M$ which is closed in the weak operator topology.

Definition 2.3. A **C^*-algebra** is a *-algebra of operators $M_0 \subseteq B(H)$ with $1_{M_0} = \mathrm{id}_H$ which is closed in the operator norm.

Remark 2.1. Since the weak operator topology is weaker than the norm topology, von Neumann algebras are C^*-algebras.

Definition 2.4. A Banach algebra is a Banach space with multiplication such that $\|xy\| \leq \|x\|\|y\|$.

We will aim to prove the following.

Theorem 2.1. *If M is a Banach algebra (with 1_M) and with an involution * satisfying $\|x^*x\| = \|x\|^2$ for all $x \in M$, then there is a injective, isometric *-algebra morphism $\theta : M \to B(H)$. In other words, any algebra satisfying these axioms is a concrete C^*-algebra.*[3]

2.2 Involutive algebras

Definition 2.5. If M is an algebra (over $\mathbb{C}$), then an **involution** on M is a map $* : M \to M$ satisfying

1. $(\lambda x)^* = \overline{\lambda} x^*$
2. $(x + y)^* = x^* + y^*$
3. $(xy)^* = y^* x^*$
4. $(x^*)^* = x$.

Example 2.1. $B(H)$ has the adjoint map as an involution.

Example 2.2. If X is compact, $C(X)$ is an algebra with the involution of complex conjugation given by $\overline{f}(x) = \overline{f(x)}$. If we take $C_0(X)$ where X is only locally compact, then we still get an algebra, but it does not have an identity.

[3]We can consider these to be the "abstract C^*-axioms."

Example 2.3. Let G be a group. Then $L^1(G)$ is an algebra with the product $f \cdot g$ of convolution. We have the involution $f^*(g) = \overline{f(g^{-1})}$.

Proposition 2.1. *The adjoint satisfies the following properties:*

1. $1^* = 1$.
2. *If x is invertible ($x \in \operatorname{Inv}(M)$), then $x^* \in \operatorname{Inv}(M)$, and $(x^*)^{-1} = (x^{-1})^*$.*

Definition 2.6. If $x = x^*$, we call x **Hermitian**. We denote the set of Hermitian elements as $M_h = \{x \in M : x = x^*\}$.

Definition 2.7. An element $x \in M$ is **normal** if $x^*x = xx^*$.

In this case, the $*$-algebra generated by x is commutative.

Definition 2.8. An element $x \in M$ is **unitary** if $x^*x = xx^* = 1$ (i.e. x is invertible an $x^{-1} = x^*$. We denote $U(M)$ to be the set of unitary elements, which is a subgroup of $\operatorname{Inv}(M)$.

Definition 2.9. An element $x \in M$ is an **isometry** if $x^*x = 1$.

Remark 2.2. In general, this does not necessarily mean that x is unitary. For example, we can take the map $x : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $(x_0, x_1, x_2, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$.

Definition 2.10. An element $x \in M$ is an **orthogonal projection** if $x^2 = x = x^*$.

Definition 2.11. An element $x \in M$ is a **partial isometry** if x^*x and xx^* are projections.

Proposition 2.2. *We can always decompose $x = \operatorname{Re} x + i \operatorname{Im}(x)$, where $\operatorname{Re} x, \operatorname{Im} x$ are Hermitian via*

$$\operatorname{Re}(x) = \frac{x + x^*}{2}, \qquad \operatorname{Im}(x) = \frac{x - x^*}{2i}.$$

Definition 2.12. If $x \in M$, the **spectrum** of x is the set $\operatorname{Spec}(x) = \{x \in \mathbb{C} : \lambda 1 - x \text{ is not invertible in } M\}$. We also call $\rho(X) = \{\lambda : \lambda 1 - x \text{ is invertible}\}$ the **resolvent** of x.

Proposition 2.3. $\operatorname{Spec}(x^*) = (\operatorname{Spec}(x))^*$, *and* $\operatorname{Spec}(x^{-1}) = (\operatorname{Spec}(x))^{-1}$.

Definition 2.13. Functionals on an involutive[4] algebra M are linear maps $\varphi : M \to \mathbb{C}$. The involution on functionals is given by $\varphi^*(x) = \overline{\varphi(x^*)}$.

[4]We call them involutive because using the term *-algebra makes people strictly think of operator algebras.

2.3 Normed involutive algebras

Definition 2.14. A **normed involutive algebra** M is an involutive algebra with a norm such that $\|xy\| \leq \|x\|\|y\|$ and $\|x^*\| = \|x\|$ for all $x \in M$. This is a **Banach algebra** if M is complete.

Definition 2.15. If M is a Banach space, we denote the **dual space** M^* to be the set of continuous linear functionals on M.

Proposition 2.4. *$\|\varphi^*\| = \|\varphi\|$ for all $\varphi \in M^*$. Also, if $\varphi = \varphi^*$, then $\|\varphi|_{M_h}\| = \|\varphi\|$.*

Notation: If X is a Banach space and $r > 0$, then we denote the closed unit ball as $(X)_r := \{x \in X : \|x\| \leq r\}$.

Definition 2.16. A Banach algebra M with involution satisfying $\|x^*x\| = \|x\|^2$ for all $x \in M$ is called an (abstract) C^*-algebra. This condition is called the **C^*-axiom**.

Remark 2.3. It is enough to show that $\|x^*x\| \geq \|x\|^2$ for all x.

Proposition 2.5. *$\|x\| = \sup_{y \in (M)_1} \|xy\|$. This gives us an isometric embedding of $M \to B(M)$ given by $x \mapsto L_x$, where $L_x(y) = xy$.*

Proposition 2.6. *If $M \neq 0$, then $\|1\| = 1$.*

Proposition 2.7. *For any $u \in U(M)$, $\|u\| = 1$.*

2.4 Spectra in Banach algebras

Definition 2.17. The **spectral radius** of x is $R(x) = \sup\{|\lambda| : \lambda \textit{ in} \operatorname{Spec}(x)\}$.

Proposition 2.8. *$R(x) \leq \|x\|$.*

If M is a Banach algebra, $x \in M$ and f is an entire function on $\mathbb{C}$, then $f(x) = \sum_{n=0}^{\infty} a - nx^n$ makes sense.

Example 2.4. We can define $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$.

Proposition 2.9. *If M has an involution and $h \in M_h$, then $\exp(ih) = \exp(-ih)$.*

Proposition 2.10. *Let M be an involutive Banach algebra, Then*

1. *If $h = h^*$, then $\exp(ih) \in U(M)$.*
2. *If $u \in U(M)$, then $\operatorname{Spec}(u) \subseteq \mathbb{T}$.*
3. *If $h = h^*$, then $\operatorname{Spec}(h) \subseteq \mathbb{R}$.*

Proof. 1. $u = \exp(ih)$ has $u^* = \exp(-ih)$ as its inverse.

2. $\mathrm{Spec}(u) = (\mathrm{Spec}(u^{-1})^{-1}$, and $\|R(u)\| \leq \|u\|$ and $\|R(u^{-1})\| \leq \|u^{-1}\|$.

3. $\mathrm{Spec}(h) = \mathrm{Spec}(h^*) = \overline{\mathrm{Spec}(h)}$. $\square$

Lemma 2.1. *Let M be a Banach algebra, and let $x \in M$ iwth $\|1 - x\| < 1$. Then x is invertible, and $\|x^{-1}\| \leq 1/(1 - \|1 - x\|)$.*

Proof. The series $y = \sum_{n=0}^{\infty}(1-x)^n$ is convergent in norm and hence makes sense in M. Then

$$xy = (1 - (1 - x)) \sum_{n=0}^{\infty} (1 - x)^n = \lim(1 - (1 - x)^{n+1}) = 1,$$

so y is an inverse for x. $\square$

Corollary 2.1. $\mathrm{Inv}(M)$ *is open, and the map* $\mathrm{Inv}(M) \to \mathrm{Inv}(M)$ *sending $x \mapsto x^{-1}$ is continuous.*

Proof. Let x be invertible, and let $\|y - x\| \leq 1/\|x^{-1}\|$. Then

$$\|x^{-1}y - 1\| \leq \|x^{-1}\|\|y - x\| < 1,$$

so $x^{-1}y$ is invertible by the lemma. So y is invertible.

Continuity follows from $x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}$. $\square$

Corollary 2.2. $\mathrm{Spec}(x) \subseteq (M)_{\|x\|}$.

Proof. If $|\lambda| > \|x\|$, then $1 > \|\lambda^{-1}x\|$, so $1 - \lambda^{-1}x$ is invertible by the lemma. So $\lambda - x$ is invertible. So $\lambda \notin \mathrm{Spec}(x)$. $\square$

Theorem 2.2. $\mathrm{Spec}(x)$ *is compact and nonempty.*

Proof. $\mathrm{Spec}(x)$ is closed by continuity of $y \mapsto y^{-1}$. It is bounded, so it is compact. To show that $\mathrm{Spec}(x) \neq \varnothing$, let $F : \rho(x) \to M$ be $F(\lambda) = (\lambda 1 - x)^{-1}$. We claim that F is analytic[5]: in fact, we have $\frac{d}{d\lambda}F(\lambda) = -(\lambda 1 - x)^{-2}$. So if $\mathrm{Spec}(x) = \varnothing$, then F is entire. But $\lim_{|\lambda| \to \infty} \|F(\lambda)\| = 0$, as

$$\|(\lambda - x)^{-1}\| = |\lambda^{-1}|\|(1 - x/\lambda)^{-1}\| \leq \frac{1}{|\lambda|}\frac{1}{1 - \|x/\lambda\|} \to 0.$$

By Liouville's theorem, F is constant, so $F = 0$.[6] But this is impossible. $\square$

Theorem 2.3 (Šilov)**.** *Let M be a Banach slagebra, and let $N \subseteq M$ be a sub Banach algebra containing 1_M. If $x \in N$, then the boundary of* $\mathrm{Spec}_N(x)$ *is a subset of of the boundary of* $\mathrm{Spec}_M(x)$.

[5] This is in the sense of holomorphic functional calculus.

[6] If you are uncomfortable with using Liouville's theorem when F is operator-valued, use this trick. Take any $\varphi \in M^*$. Then $\lambda \mapsto \varphi(F(\lambda))$ is analytic, entire, and $= 0$. Using Hahn-Banach, it follows that $F = 0$.

Remark 2.4. We always have $\operatorname{Spec}_M(x) \subseteq \operatorname{Spec}_N(x)$. This theorem gives part of the other direction.

Proof. It suffices to show that the boundary of $\operatorname{Spec}_N(x)$ is contained in $\operatorname{Spec}_M(x)$. Let $\lambda_0 \in \partial \operatorname{Spec}_N(x)$, and let $\{\lambda_n\} \subseteq \rho_N(x)$ with $\lambda_n \to \lambda_0$. If for some n, m, we were to have $\|(\lambda^n - x)^{-1}\| < 1/|\lambda_0 - \lambda_n|$, it would follow that $\|(\lambda_0 - x) - (\lambda_n - x)\| < 1/\|(\lambda_n - x)^{-1}\|$. Thus, $\lambda_0 - x$ is invertible in N by the lemma. This is a contradiction, so $\|(\lambda_n - x)^{-1}\| \to \infty$. Now if $\lambda_0 \notin \operatorname{Spec}_M(x)$, then $\|(\lambda - x)^{-1}\|$ is bounded for λ close enough to λ_0. This contradicts $\|(\lambda_n - x)^{-1}\| \to \infty$. $\square$

Lemma 2.2 (Spectral radius formula)**.** *$R(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.*

We will prove this later.

2.5 Contractivity of morphisms into C^*-algebras

Proposition 2.11. *If $N \subseteq M$ are C^*-algebras with $1_M \in N$ and $x \in N$, then $\operatorname{Spec}_N(x) = \operatorname{Spec}_M(x)$.*

Proof. Assume first that $x = x^*$. Then $\operatorname{Spec}_N(x), \operatorname{Spec}_M(x) \subseteq \mathbb{R}$. Then Šilov's theorem implies that $\operatorname{Spec}_N(x) = \operatorname{Spec}_M(x)$. For general x, invertibility of x in M implies invertibility of x^*x in M. This implies that x^*x is invertible in N, which provides a $y \in N$ such that $(yx^*)x = 1$. So x is invertible in N. $\square$

Proposition 2.12. *Let M be a Banach involutive algebra, and let N be a C^*-algebra. Let $\pi : M \to N$ be a unital *-morphism.*[7]

Proof. For $y \in N$ with $y = y^*$, we have $\|y^2\| = \|y^*y\| = \|y\|^2$. Iterating thism we get $\|y^{2^n}\|^{1/2^n} = \|y\|$. The left hand side tends to $R(y)$, so $R(y) = \|y\|$. If $x \in M$, we have $\operatorname{Spec}_N(\pi(x)) \subseteq \operatorname{Spec}_M(x)$ (since π is an algebra homomorphism, it preserves invertibility). So we get $R_N(\pi(x)) \leq R_M(x) \leq \|x\|$. So

$$\|\pi(x)\|^2 = \|\pi(x^*x)\| = R_N(\pi(x^*x)) \leq \|x^*x\| \leq \|x\|^2.$$

$\square$

[7]This means it is an algebra homomorphism. This condition says nothing about the norm, a priori.

3 The Spectral Radius Formula And The Gelfand Transform

3.1 Characters of Banach algebras

Last time, we used the following result to show that morphisms to C^*-algebras are contractive.

Lemma 3.1 (Spectral radius formula). $R(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

This is really a result about commutative Banach algebras, so to prove it we will discuss the commutative case.

Definition 3.1. Let M be a Banach algebra (with 1_M). A **character** on M is a nonzero linear $\varphi : M \to \mathbb{C}$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in M$. We denote X_M as the space of all characters on M.

Proposition 3.1. *Let M be a Banach algebra. Any $\varphi \in X_M$ is automatically continuous with $\|\varphi\| = 1$.*

Proof. For any $x \in M$, $x - \varphi(x) \cdot 1 \in \ker(\varphi)$. Write $x = (x - \varphi(x) \cdot 1) + \varphi(x) \cdot 1$. Then

$$\|\varphi\| = \sup_{x\in(M)_1} |\varphi(x)| = \sup_{\substack{y\in\ker(\varphi)\\ \lambda\neq 0}} \frac{|\varphi(y + \lambda \cdot 1)|}{\|y + \lambda \cdot 1\|} = \sup_{\substack{y\in\ker(\varphi)\\ \lambda\neq 0}} \frac{1}{\|(y/\lambda) + 1\|}.$$

If $\|y' + 1\| < 1$, then y' is invertible, which means $y' \notin \ker(\varphi)$. So this equals 1. $\square$

Corollary 3.1. *$X_M \subseteq (M^*)_1$ is $\sigma(M^*, M)$-compact (weak* compact).*

Proof. X_M is closed in the weak* topology. $\square$

3.2 The Gelfand transform

Definition 3.2. The **Gelfand transform** $\Gamma : M \to C(X_M)$ is given by $\Gamma(x)(\varphi) := \varphi(x)$.

Proposition 3.2. *The Gelfand transform has the following properties:*

1. *Γ is an algebra morphism.*
2. *$\|\Gamma(x)\|_\infty \le \|x\|$.*

Theorem 3.1. *If M is a Banach algebra such that any $x \neq 0$ is invertible (a division algebra), then $M = \mathbb{C}$.*

Proof. If $x \in M$, then $\mathrm{Spec}_M(x) \neq \varnothing$, so let $\lambda_x \in \mathrm{Spec}_M(x)$. Then $x - \lambda_x 1$ is not invertible, so $x - \lambda_x = 0$. So $\lambda_x 1 = x$. $\square$

Proposition 3.3. *If M is a Banach algebra and $J \subseteq M$ is a closed, 2-sided ideal, then M/J has a Banach algebra structure given by $\|x + J\| = \inf_{y \in J} \|x + y\|$.*

Proposition 3.4. *If M is a commutative Banach algebra, then there is a correspondence between X_M and the space of maximal, 2-sided ideals of M given by $\varphi \mapsto \ker(\varphi)$.*

Proof. Let $\varphi \in X_M$, and let J be an ideal such that $\ker(\varphi) \subsetneq J$. Let $x \in J \setminus \ker(\varphi)$. Then $x = (x - \varphi(x) \cdot 1) + \varphi(x) \cdot 1$, so 1 is in the span of x and $\ker(\varphi)$, which is contained in J. So J is an ideal containing M and hence equals M. That is, $\ker(\varphi)$ is maximal.

If J is a maximal ideal in M, then $\overline{J}$ is an ideal (using the $\|1 - x\| < 1 \implies x$ is invertible lemma), so J is closed. Then let $\varphi_J : M \to M/J$ be the natural projection map. But since J is maximal, M/J is a division algebra. So $M/J = \mathbb{C}$. This means $J = \ker(\varphi_J)$, where φ_J is a character. $\square$

Proposition 3.5. *If M is a commutative Banach alagebra, then $X_M = \varnothing$ and $x \in M$ is invertible if and only if $\Gamma(x)$ is invertible.*

Proof. If $x \in M$ is invertible, then $\Gamma(x^{-1})$ is the inverse of $\Gamma(x)$. If $x \in M$ is not invertible, then xM is a proper, 2-sided ideal in M. Let $J \subseteq M$ be a maximal 2-sided ideal containing xM. Then $\varphi_J(x) = 0$, so $\Gamma(x)$ is not invertible. $\square$

We can summarize our results in the following theorem.

Theorem 3.2. *Let X be a commutative Banach algebra.*

1. $X_M \neq \varnothing$.
2. Γ *is an algebra homomorphism.*
3. $\|\Gamma(x)\|_\infty \leq \|x\|$ *for all* $x \in M$.
4. $x \in \mathrm{Inv}(M) \iff \Gamma(x) \in \mathrm{Inv}(C(X_M))$.

3.3 The spectral mapping theorem and the spectral radius formula

Corollary 3.2. *Let M be a commutative Banach algebra, and let $x \in M$. Then* $\mathrm{Spec}_M(x) = \mathrm{Ran}(R(x))) = \mathrm{Spec}_{C(X_M)}(\Gamma(x))$. *Thus,* $R_M(x) = \|\Gamma(x)\|_\infty$.

Proof.

$$\begin{aligned} \lambda \notin \mathrm{Spec}_M(x) &\iff \lambda - x \text{ is invertible in } M \\ &\iff \lambda - \Gamma(x) \text{ is invertible in } C(X_M) \\ &\iff \lambda \notin \mathrm{Range}(\Gamma(x)). \end{aligned}$$ $\square$

Corollary 3.3 (Spectral mapping theorem)**.** *Let M be a Banach algebra, let $x \in M$, and let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $f(z) = \sum_{n=0}^\infty a_n z^n$. Then* $\mathrm{Spec}_M(f(x)) = f(\mathrm{Spec}(x))$.

Remark 3.1. The function $f(x)$ makes sense, as the sum is absolutely convergent in norm. The radius of convergence is $(\limsup |a_n|^{1/n})^{-1} = \infty)$.

We can now prove the spectral radius formula.

Proof. Let M_0 be the Banach algebra generated by $1, x, f(x), (x-\lambda)^{-1}$ for all $\lambda \in \rho_M(x)$, and $(f(x)-\mu)^{-1}$ for all $\mu \in \rho_M(f(x))$, where ρ denotes the resolvent. Then M_0 is commutative, so $\mathrm{Spec}_{M_0}(x) = \mathrm{Spec}_M(x)$ and $\mathrm{Spec}_{M_0}(f(x)) = \mathrm{Spec}_M(f(x))$. So we may assume that M is commutative.

From the corollary, we have $\mathrm{Spec}_M(x^n) = (\mathrm{Spec}_M(x))^n$ (using the Gelfand transform). So $R_M(x)^n = R_M(x^n) \leq \|x^n\|$. We get that $R_M(x) \leq \liminf_n \|x^n\|^{1/n}$. Let $G(\lambda) = -\lambda \sum_{n=0}^\infty \frac{x^n}{\lambda^n}$. This sum converges absolutely for $|\lambda| > \|x\|$ and converges to $(x-\lambda)^{-1}$. But for $|\lambda| > R_M(x)$ and $\varphi \in M^*$, $\varphi((x-\lambda)^{-1})$ is analytic, and $\lambda \mapsto \varphi(G(\lambda))$ is analytic and agrees with $\varphi((x-\lambda)^{-1})$ there. So we conclude that for every $\varphi \in M^*$, $\lim_{n\to\infty} \varphi(\lambda^{1-n}x^n) = 0$ whenever $|\lambda| > R_M(x)$.

Apply the uniform boundedness principle to $\lambda^{1-n}x^n \in M$, viewed as an element of M^{**}. So there exists $K(\lambda) > 0$ such that $\|\lambda^{1-n}x^n\| \leq K(\lambda)$ for all n. So

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq \limsup_{n\to\infty} K(\lambda)^{1/n} |\lambda|^{(n-1)/n} = |\lambda|$$

for each $|\lambda| > R_M(x)$. $\square$

Corollary 3.4. *Let M be a commutative Banach algebra. Then the Gelfand transform $\Gamma : M \to C(X_M)$ is an isometry if and only if $\|x^2\| = \|x\|^2$ for all $x \in M$.*

Proof. Suppose Γ is an isometry. We have $R(x)^2 = R(\Gamma(x))^2 = \|\Gamma(x)\|^2$, and $R(x^2) = R(\Gamma(x^2)) = \|\Gamma(x^2)\|$. These are equal, so $\|x\| = \|\Gamma(x)\| \implies \|x^2\| = \|x\|^2$.

Conversely if $\|x^2\| = \|x\|^2$, then $\|x\| = R(x)$ by the spectral radius formula (we did this argument before). $\square$

3.4 Characterization of commutative C^*-algebras

Recall the Stone-Weierstrass theorem.

Theorem 3.3 (Stone-Weierstrass)**.** *Let X be compact, and let $M \subseteq C(X)$ be a norm-closed, *-closed subalgebra with $1 \in M$ that separates points (i.e. for all $t_1 \neq t_2 \in X$, there is an $f \in M$ such that $f(t_1) \neq f(t_2)$). Then $M = C(X)$.*

Theorem 3.4 (Gelfand)**.** *Let M be a commutative C^*-algebra.*

1. *If $\varphi \in X_M$, then $\varphi = \varphi^*$; i.e. $\varphi(x^*) = \varphi(x)^*$ for all x.*

2. *$\Gamma : M \to C(X)$ is a *-algebra isometric isomorphism.*

Proof. If $x = x^* \in M$, then $\varphi(x) \in \mathrm{Spec}_M(x) \subseteq \mathbb{R}$.

By the first part, $\Gamma(M)$ is *-closed. By definition $\Gamma(M)$ separates points: $\varphi_1 \neq \varphi_2$ means that there is an x usch that $\varphi_1(x) \neq \varphi_2(x)$. By the Stone-Weierstrass, $\overline{\Gamma(M)} = X$. By the C^*-algebra axiom, $\|x^2\| = \|x\|^2$, so Γ is isometric. □

3.5 Continuous functional calculus

Lemma 3.2. *Let M be a commutative C^*-algebra. If $x \in M$ and M is generated by x, then $X_M \simeq \mathrm{Spec}(x)$ via $\varphi \mapsto \varphi(x)$.*

Example 3.1. Let $T \in B(H)$ be normal ($T^*T = TT^*$). Then the spectrum of $C^*(\{T\})$ can be identified with $\mathrm{Spec}(T)$.

So if M is a C^*-algebra, $x \in M$ is normal, and $f \in C(\mathrm{Spec}(x))$, we can think of $f(x) \in M$ by $f(x) = \Gamma^{-1}(f)$. This is **continuous functional calculus**.

4 Correspondence Between Homeomorphisms and C^*-Algebra Morphisms

4.1 Recap: Homeomorphism between X_M and $\operatorname{Spec}(x)$

Recall our results from last time.

Proposition 4.1. *Let M be a commutative Banach algebra (over $\mathbb{C}$). Then $\operatorname{Spec}_M(x) = \operatorname{Spec}_{C(X_M)}(\Gamma(x))$.*

Proposition 4.2. *Let M be a C^*-algebra, and let $M_0 \subseteq M$ be a sub C^*-algebra. Then $\operatorname{Spec}_M(x) = \operatorname{Spec}_{M_0}(x)$.*

Theorem 4.1 (Gelfand)**.** *Let M be a commutative C^*-algebra.*

1. *If $\varphi \in X_M$ is a character, then $\|\varphi\| = 1$ and $\varphi = \varphi^*$.*
2. *$\Gamma : M \to C(X_M)$ is a $*$-algebra isomorphism.*

Proposition 4.3. *Let M be a C^*-algebra generated by $x \in M$ and 1.*[8] *Then $\Psi : X_M \simeq \operatorname{Spec}(x)$ via $\varphi \mapsto \varphi(x)$ is a homeomorphism of compact spaces.*

Remark 4.1. Note that $\varphi(x) = \Gamma(x)(\varphi)$.

Proof. The map is surjective and well-defined by the first proposition above. Also, Ψ is continuous. If $\Psi(\varphi_1) = \Psi(\varphi_2)$, then $\varphi_1(x) = \varphi_2(x)$. But this implies that $\varphi_1(x^*) = \varphi_2(x^*)$. So $\varphi_1 = \varphi_2$ on all of M, as x generates M. So Ψ is injective. □

4.2 Correspondence between homeomorphisms and C^*-algebra morphisms

Remark 4.2. If $\Delta : Z \to Y$ is a map between compact spaces, then we get a map $\Delta^* : C(Y) \to C(Z)$ given by $\Delta^*(f) = f \circ \Delta$. The map Δ^* is a a $*$-algebra homomorphism.

Conversely, if $\theta : M \to N$ is a morphism of unital C^*-algebras, we can view $\theta : C(X_M) \to C(X_N)$. Then there is a canonical $\Delta : X_N \to X_M$ such that $\theta = \Delta^*$ as follows. If $\varphi : N \to \mathbb{C}$ is multiplicative, then $\varphi \circ \theta : M \to \mathbb{C}$ is multiplicative. So $\Delta(\varphi) = \varphi \circ \theta$ is a well-defined map $X_M \to X_N$. Then $\Delta^* = \theta$. We denote this Δ by θ_* (so $(\theta_*)^* = \theta$).

Moreover, θ is surjective if and only if θ_* is injective and is injective if and only if θ_* is surjective. Thus, θ is an C^*-algebra isomorphism if and only if θ_* is a homeomorphism.

This is very important! It says that any homeomorphism between compact spaces corresponds to a *-algebra morphism between C^*-algebras.

Proposition 4.4. *If θ is surjective, then θ_* is injective.*

[8] Alternatively, we can say, "Let M_0 be the sub C^*-algebra of M generated by x and 1.

Proof. Let $\varphi_1 \neq \varphi_2 \in X_N$. Then $\varphi_1 \circ \theta \neq \varphi_2 \circ \theta$. $\square$

Proposition 4.5. *If θ is injective, then θ_* is surjective.*

Proof. We get that $\theta : M \to N$ is isometric, as $\|y^*y\|_M = \mathrm{Spec}_M(y^*y) = \mathrm{Spec}_N(y^*y) = \|y^*y\|_N$ since y^*y is self-adjoint; then the C^*-condition gives that $\|y\|_M = \|y\|_N$.

Take a $\varphi \in X_M$ and consider the corresponding maximal ideal $M_\varphi \subseteq M$. Then $N\theta(M_\varphi)$ is a closed proper ideal in N (proper because it does not contain 1). Any maximal ideal M' containing $N\theta(M_\varphi)$ has the property that its character $\varphi' = \varphi_{M'} \in X_N$ satisfies $\theta_*(\varphi') = \varphi$. $\square$

4.3 Continuous functional calculus

Let's be a bit more clear about a point made last lecture, using this viewpoint we have established.

Remark 4.3. Let M be a commutative C^*-algebra generated by x (so x is normal). Then $M \simeq C(X_M)$ via Γ. Note that since $\varphi(x) = \Gamma(x)(\varphi)$, using this identification, the map $(\Psi^{-1}) * \circ\Gamma$ sends $x^n \mapsto (z \mapsto z^n)$ and $(x^*)^m \mapsto (\overline{z} \mapsto \overline{z}^m)$. So for $f \in C(\mathrm{Spec}(x))$, we can define $f(x) := ((\Psi^{-1}) * \circ\Gamma)^{-1}(f)$. This is called **continuous functional calculus** for normal elements in a C^*-algebra.

5 The GNS Construction

5.1 The idea: turning abstract C^*-algebras in to concrete ones

Let M be a C^*-algebra (with 1_M) and $x = x^* \in M_h$. Then the C^*-algebra generated by x can be identified with $\mathrm{Spec}(x)$. Then denote

$$x_+ = f_+(x), \qquad x_- = f_-(x),$$

where

$$f_+(x) = \max\{x, 0\}, \qquad f_-(x) = -\min\{x, 0\}.$$

Then $x = x_+ - x_-$, $x_+ x_- = 0$, and we can also define $|x| = x_+ + x_- = (x^2)^{1/2}$. If $\mathrm{Spec}(x) \subseteq [0, \infty)$, then we can define $\sqrt{x}$ using functional calculus.

Lemma 5.1. *If $x = x^* \in M$ and $\|1 - x\| \leq 1$, then $\mathrm{Spec}(x) \subseteq [0, \infty)$. Conversely, if $\mathrm{Spec}(x) \subseteq [0, \infty)$, then $\|1 - x\| \leq 1$.*

Proof. This follows from functional calculus. $\square$

Lemma 5.2. *Let S, T be elements of a Banach algebra. Then $\mathrm{Spec}(ST) \cup \{0\} = \mathrm{Spec}(TS) \cup \{0\}$.*

Proof. If $\lambda \neq 0$ and $TS - \lambda 1$ has inverse u, then $TSu = \lambda u + 1$, so

$$(ST - \lambda 1)(SuT - 1) = STSuT - ST - \lambda - \lambda SuT + \lambda 1 = \lambda 1. \qquad \square$$

Recall: We want to show that if M is a C^*-algebra, there is an isometric embedding $\pi : M \to B(H)$, where H is a Hilbert space; that is, every abstract C^*-algebra is a concrete C^*-algebra. To get the isometry property, we only need $\|\pi(x)\|^2 = \|x\|^2$, which means we need $\|\pi(x^*x)\| = \|x^*x\|$. This is the spectral radius of x^*x and $\pi(x^*x)$, so we need only show that π is injective.

Suppose we have that if $x \neq 0$ then there is a $\pi_x : M \to B(H_x)$ with $\pi_x(x) \neq 0$. Then we can take $\bigoplus_x \pi_x : M \to \bigoplus B(H_x)$. So we only need to find π_x for each x. To find π_x, we claim that all we need is a functional φ which has $\varphi(y^*y) \geq 0$ for $y \in M$ and $\varphi(x^*x) \neq 0$. Then we will be able to get a Hilbert space by looking at M itself with the inner product $\langle y, x \rangle_\varphi = \varphi(y^*x)$ (this is a Hilbert space if we mod out by some equivalence relation). To find a functional φ, we will need to use Hahn-Banach.

5.2 Characterizing positive elements in a C^*-algebra

Proposition 5.1 (Positive elements in C^*-algebras)**.** *Let M be a C^*-algebra, and let $x = x^* \in M_h$. The following are equivalent:*

1. $\mathrm{Spec}(x) \subseteq [0, \infty)$.

2. $x = y^*y$ *for some* $y \in M$.

3. $x = h^2$ *for some* $h \in M_h$.

Also, if we denote M_+ *to be the set of elements satisfying these conditions, then* M_+ *is a closed, convex cone in* M_h *(*$x \in M_+, \lambda \geq 0 \implies \lambda x \in M_+$ *and* $x, y \in M_+ \implies x + y \in M_+$*). Moreover,* $M_+ \cap (-M_+) = \{0\}$.

Proof. Let P be the set of elements in M_h satisfying condition 1.

(1) $\implies$ (3): Take $h = \sqrt{x}$ by functional calculus.

(3) $\implies$ (2): Take $y = y^* = h$.

(3) $\implies$ (1): Since h is self-adjoint, $\mathrm{Spec}(h) \subseteq \mathbb{R}$. Then we have $\mathrm{Spec}(h^2) = (\mathrm{Spec}(h))^2 \subseteq [0, \infty)$.

(2) $\implies$ (3): Write $y^*y = (y^*y)_+ - (y^*y)_- := u^2 - v^2$. Then

$$(yv)^*(yv) = v(y^*y)v = v(u^2 - v^2)v = -v^4$$

has spectrum $\subseteq (-\infty, 0]$. Let $yv = s + it$ with $s, t \in M_h$. Then

$$(yv)(yv)^* = \underbrace{(s - it)(s + it)}_{s^2+t^2} + \underbrace{(s + it)(s - it)}_{s^2+t^2},$$

so if P is a convex cone, then this is in P. Then also $(yv)^*(yv) \in P$ (because $\mathrm{Spec}(TS) \cup \{0\} = \mathrm{Spec}(ST) \cup \{0\}$). So we get that $\mathrm{Spec}((yv)^*(yv)) = 0$, which means that $yv = 0$. So $v = 0$.

To show that P is a cone, we use that for $x \in M_h$, $x \in P \iff \|\|x\|(1 - x)\| \leq 1$ (from the lemma before). This implies that P is closed. On the other hand, if $x \in P$ and $\lambda > 0$, then $\lambda x \in P$ by functional calculus. And if $x, y \in P$ (and now we can assume $\|x\|, \|y\| \leq 1$), then

$$\left\| 1 - \frac{x + y}{2} \right\| \leq \frac{1}{2} \underbrace{\|1 - x\|}_{\leq 1} + \frac{1}{2} \underbrace{\|1 - y\|}_{\leq 1} \leq 1,$$

so $\frac{x+y}{2} \in P$. It follows that P is a closed, convex cone. This completes the proof. $\square$

So from now on, if M is a C^*-algebra, then we denote M_+ to be the cone of positive elements.

5.3 Positive linear functionals

Definition 5.1. A functional $\varphi : M \to \mathbb{C}$ on an involutive algebra is **positive** if $\varphi(x^*x) \geq 0$ for all $x \in M$ and $\varphi(M_+) \subseteq [0, \infty)$.

Definition 5.2. A **state**[9] on an involutive Banach algebra is a positive continuous functional with $\|\varphi\| = 1$.

[9] This terminology comes from physics.

Proposition 5.2. *If M is an involutive algebra and φ is a positive functional, then M has a pre-Hilbert space structure H_φ with the pre-inner product $\langle x, y\rangle_\varphi = \varphi(y^*x)$.*

Corollary 5.1. *Let $I_\varphi := \{x \in M : \varphi(x^*x) = 0\}$. M/I_φ is an inner product space with $\langle \cdot, \cdot \rangle_\varphi$. The completion is a Hilbert space.*

Proof. We just need I_φ to be a vector space. We have the Cauchy-Schwarz inequality: we have for all $\lambda \in \mathbb{C}$, $\langle x + \lambda y, x + \lambda y\rangle \geq 0$, so the discriminant is ≤ 0; this translates into the desired inequality. Now I_φ is a vector space because we have $\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle x + i^k y, x + i^k y\right\rangle_\varphi$. □

Proposition 5.3. *I_φ is a left M-ideal. That is, if $y \in I_\varphi$, then $xy \in I_\varphi$ for any $x \in M$.*

Lemma 5.3. *If M is a Banach algebra and $x \in (M)_1$, with $x = 1 + x'$, then the series*

$$h = 1 + \frac{1}{1!}\cdot\frac{1}{2}x' + \frac{1}{2!}\cdot\frac{1}{2}\left(\frac{1}{2} - 1\right)(x')^2 + \cdots + \frac{1}{n!}\left(\frac{1}{2} - 1\right)\cdots\left(\frac{1}{2} - (n-1)\right)(x')^n + \cdots$$

is absolutely convergent with $h^2 = x = 1 + x'$. Moreover, if x is self-adjoint, then so is h.

Proposition 5.4. *If M is an involutive Banach algebra and φ is positive on M, then φ is continuous and $\|\varphi\| = \varphi(1)$.*

Proof. By Cauchy-Schwarz, $|\varphi(1x)|^2 \leq \varphi(1)\varphi(x^*x)$. If $y = y^*$ and $\|y\| \leq 1$, then, by the lemma, we have $1 - y = h^2$ with $h = h^*$. Given this representation, $\varphi(x^*x) \leq \varphi(1)$, so $|\varphi(x)| \leq \varphi(1)$. □

Corollary 5.2. *If M is an involutive Banach algebra with 1_M, then the space of states $S(M)$ is convex and weakly compact in $(M^*)_1$.*

Proposition 5.5. *Let M be an involutive Banach algebra, and let φ be positive. Then for all $x, y \in M$,*

$$|\varphi(y^*xy)| \leq \|x\|\varphi(y^*y).$$

Proof. The functional $\varphi_y(x) := \varphi(y^*xy)$ is positive. So $|\varphi_x(x)| \leq \varphi_y(1)\|x\|$. We then have $|\varphi(y^*xy)| \leq \varphi(y^*y)\|x\|$. □

5.4 The GNS construction

Corollary 5.3 (GNS[10] construction)**.** *Let M be an involutive Banach algebra, and let φ be positive. Then $\pi_\varphi : M \to \mathcal{B}(M/I)$ given by $\pi_\varphi(x)(\hat{y}) = \widehat{xy}$ is an isometric *-isomorphism of algebras.*

[10] This is Gelfand, Naimark, and Segal. Gelfand and Naimark only proved it for the commutative case.

Proof. We have

$$\|\pi_\varphi(x)(\hat{y})\|^2 = \varphi(y^*x^*xy) \leq \|x^*x\|\varphi(y^*y) \leq \|x\|^2\|\hat{y}\|_{M/I_\varphi}.$$

So $\|\pi_\varphi(x)\| \leq \|x\|$, so $\pi_\varphi(x)$ is continuous and extends to all of M/I_φ. We also have

$$\pi_\varphi(x_1x_2) = \pi_\varphi(x_1)\pi_\varphi(x_2),$$

$$\pi_\varphi(x^*) = \pi_\varphi(x)^*. \qquad \square$$

Such a map π_φ is called a **representation**.

Proposition 5.6. *If M is a C^*-algebra, then φ is positive if and only if $\|\varphi\| = \varphi(1)$.*

Proof. ($\implies$): We have already shown this.

($\impliedby$): If $\varphi(1) = \|\varphi\| = 1$ and $x \geq 0$ in M, suppose $\varphi(x) \not\geq 0$. Then there exists a disc $D \subseteq \mathbb{C}$ centered at some $z_0 \in \mathbb{C}$ such that $\mathrm{Spec}(x) \subseteq D$ but $\varphi(x) \notin D$. Thus, $\mathrm{Spec}(x - z_0 1) \subseteq B_R(0)$, and $x - z_0 1$ is normal. So $\|x - z_0 1\| \leq R$, and

$$|\varphi(x) - z_0| = |\varphi(x - z_0)| \leq \|\varphi\|\|x - z_0\| \leq \|x - z_0\| \leq R.$$

This is a contradiction. $\square$

It remains to show that we can find enough positive linear functionals. We will finish this next time.

6 GNS Construction and Topologies on $\mathcal{B}(H)$

6.1 Every C^*-algebra is an operator algebra

Recall the GNS construction: Let M be a C^* algebra, and let φ be a positive functional. We get a Hilbert space H_φ with $\langle x, y\rangle_\varphi = \varphi(y^*x)$ and a representation $\pi_\varphi(x)(\hat{y}) = \widehat{xy}$.

Remark 6.1. Our representation uses left multiplication, but there is no preference. If we take $\langle x, y\rangle = \varphi(xy^*)$, then we could use right multiplication for our representation π.

This construction gives us $\xi_\varphi = \hat{1}_M$. This gives $H_\varphi = \overline{s\pi_\varphi(M)\xi_\varphi}$. We call ξ_φ a **cyclic vector** for the representation π_φ.

Lemma 6.1. *If H_i is a Hilbert space for all i and $T_i \in B(H_i)$ with $\sup \|T_i\| < \infty$, then $\bigoplus_i T_i \in \mathcal{B}(H_i)$, where $(\bigoplus_i T_i)(\bigoplus_i \xi_i) = \bigoplus_i T(\xi_i)$.*

Theorem 6.1 (GNS)**.** *If M is a C^*-algebra, there exists an isometric, unital, *-algebra morphism $\pi : M \to \mathcal{B}(H)$, where H is a Hilbert space.*

Proof. If $\pi_i : M \to B(H_i)$ are representations for all i, we can define $\pi = \bigoplus_i \pi_i : M \to \mathcal{B}(\bigoplus_i H_i)$ by $\pi(x) = \bigoplus_i \pi_i(x)$. The inner product on $\bigoplus_i H_i$ is $\langle (\xi_i)_i, (\eta_i)_i\rangle = \sum_i \langle \xi_i, \eta_i\rangle_{H_i}$. Injective implies isometric, so it suffices to find a 1 to 1 representation. So it suffices to show that for any $x \neq 0$ in M, there exists $\pi_x : M \to B(H_x)$ such that $\pi_x(x) \neq 0$. By the GNS construction, it suffices to get a positive functional φ on M such that $\|\pi_\varphi(x)\hat{1}\| = \varphi(x^*x) \neq 0$.

The subspace M_+ is closed and convex in M_h and does not contain $-x^*x$. By Hahn-Banach, there exists a continuous $\varphi : M_h \to \mathbb{R}$ and an $\alpha \in \mathbb{R}$ such that $\varphi(M_+) < [\alpha, \infty)$ and $\varphi(-x^*x) < \alpha$. Since $0 \in M_+$, $0 \in [\alpha, \infty)$, making $\alpha \leq 0$. If $\alpha > 0$, then $\lambda y \in M_+ \implies \varphi(\lambda y) < 0$. This is a contradiction, so $\alpha = 0$. So φ is positive and $\varphi(x^*x) \neq 0$. $\square$

Remark 6.2. To get the isometry property, we could have produced a φ such that $\varphi(x^*x) = \|x^*x\|$.

6.2 Topologies on $\mathcal{B}(H)$

If H is a Hilbert space, we have multiple choices for norms on $\mathcal{B}(H)$.

Definition 6.1. The **operator norm topology** is the norm topology given by

$$\|T\| = \sup_{\xi \in (H)_1} \|T\xi\| = \sup_{\xi, \eta \in (H)_1} |\langle T\xi, \eta\rangle|.$$

Definition 6.2. The **weak operator topology** is the topology generated by the semi-norms $T \mapsto |\langle T\xi, \eta\rangle|$ for all $\xi, \eta \in H$.

Definition 6.3. The **strong operator topology** is the topology generated by the seminorms $T \mapsto \|T\xi\|$ for all $\xi \in H$.

The WOT is weaker than the SOT, which is weaker than the NT.

Definition 6.4. A **von Neumann algebra** is a *-algebra $M \subseteq B(H)$ with $1_M = \mathrm{id}_M \in M$ which is closed in the weak operator topology.

So every von Neumann algebra is a C^*-algebra.

Definition 6.5. Let X be a Banach space, and let $Y \subseteq X^*$ be a vector subspace. The $\sigma(X,Y)$ **topology** on X is given by the seminorms $x \mapsto |\varphi(x)|$ for $\varphi \in Y$.

Proposition 6.1. *Let X be a Banach space, and let $Y \subseteq X^*$ be a vector subspace.*

1. *A linear functional $\varphi : X \to \mathbb{C}$ is $\sigma(X,Y)$-continuous if and only if $\varphi \in Y$.*
2. *A linear functional $\varphi : X \to \mathbb{C}$ is $\sigma(X,Y)$-continuous on $(X)_1$ iff $\varphi \in \overline{Y} \subseteq X^*$ (closure with respect to the norm topology).*
3. *The topologies $\sigma(X,Y)$ and $\sigma(X,\overline{Y})$ coincide on $(X)_1$.*
4. *If $\overline{Y} = Y$, a linear functional φ is $\sigma(X,Y)$-continuous if and only if it is $\sigma(X,Y)$-continuous on $(X)_1$.*

Denote by $B_\sim = \mathrm{span}\{\omega_{\cdot\xi,\eta} = \langle \xi, \eta\rangle : \xi, \eta \in H\} \subseteq \mathcal{B}^*$, and denote $\mathcal{B}_* = \overline{\mathcal{B}_\sim} \subseteq \mathcal{B}^*$.

Remark 6.3. The weak operator topology is the $\sigma(\mathcal{B}, \mathcal{B}_\sim)$ topology on $\mathcal{B}(H)$.

Remark 6.4. Let $FR \subseteq \mathcal{B}(H)$ be the space of finite rank operators. Then $FR \to \mathcal{B}_\sim$ given by $T \mapsto \omega_T$, where $\omega_T(x) = \mathrm{tr}_{\mathcal{B}(H)}(xT)$ is an isomorphism.

Definition 6.6. The **ultraweak toplogy** on $\mathcal{B}(H)$ is the $\sigma(\mathcal{B}, \mathcal{B}_*)$ topology.

Corollary 6.1. *Let $\mathcal{B} = \mathcal{B}(H)$ for a Hilbert space H.*

1. *$\mathcal{B}_\sim$ is the space of weak operator continuous functionals on $\mathcal{B}(H)$.*
2. *$\mathcal{B}_*$ is the space of ultraweak continuous functionals functionals on $\mathcal{B}$*
3. *$\varphi : B \to \mathbb{C}$ is ultraweak continuous if and only if it is weak operator continuous on $(\mathcal{B})_1$.*
4. *The weak and ultraweak topologies coincide on $(\mathcal{B})_1$.*

Theorem 6.2. *$\varphi : \mathcal{B} \to \mathbb{C}$ is weak operator continuous if and only if it is it is strong operator continuous.*

7 WO and SO Continuity of Linear Functionals and The Pre-Dual of $\mathcal{B}$

7.1 Weak operator and strong operator continuity of linear functionals

Lemma 7.1. *Let X be a vector space with seminorms $p_1, \dots, p_n$. Let $\varphi : X \to \mathbb{C}$ be a linear functional such that $|\varphi(x)| \leq \sum_{i=1}^n p_i(x)$ for all $x \in X$. Then there exist linear functionals $\varphi_1, \dots, \varphi_n : X \to \mathbb{C}$ such that $\varphi = \sum_i \varphi_i$ with $|\varphi_i(x)| \leq p_i(x)$ for all $x \in X$ and for all i.*

Proof. Let $D = \{\tilde{x} = (x, \dots, x) : x \in X\} \subseteq X^n$, which is a vector subspace. On X^n, take $p((x_i)_{i=1}^n) = \sum_i p_i(x_i)$. We also have a linear map $\tilde{\varphi} : D \to \mathbb{C}$ given by $\tilde{\varphi}(\tilde{x}) = \varphi(x)$. This map satisfies $|\tilde{(}\tilde{x})| \leq p(\tilde{x})$. By the Hahn-Banach theorem, there exists an extension $\psi \in (X^n)^*$ of $\tilde{\varphi}$ such that $|\psi(x_1, \dots, x_n)| \leq p(x_1, \dots, x_n)$. Now define $\varphi_k(x) := \psi(0, \dots, x, 0, \dots)$, where the x is in the k-th position. $\square$

Theorem 7.1. *Let $\varphi : \mathcal{B} \to \mathbb{C}$ be linear. φ is weak operator continuous if and only if it is it is strong operator continuous.*

Proof. We only need to show that if φ is strong operator continuous, then it is weak operator continuous. So assume there exist $\xi_1, \dots, \xi_n \in X$ such that $|\varphi(x)| \leq \sum_{i=1}^n \|x\xi_i\|$ for all $x \in \mathcal{B}$. By the lemma, we can split $\varphi = \sum \varphi_k$, such that $|\varphi_k(x)| \leq \|x\xi_k\|$ for all x and k. By the Riesz representation theorem, there exists an $\eta_k \in H$ such taht $\varphi_k(x) = \langle x\xi_k, \eta_k \rangle$. So $\varphi(x) = \sum_k \langle x\xi_k, \eta_k \rangle$. So φ is weak operator continuous. $\square$

Corollary 7.1. *Any closure in $\mathcal{B}(H)$ of a convex set is the same with respect to the weak operator and strong operator topologies.*

Proof. If we have a point in the closure wrt one topology but not in the other, we can separate it with a hyperplane using the geometric Hahn-Banach theorem. $\square$

Corollary 7.2. *Let $M \subseteq \mathcal{B}(H)$ be a vector subspace. Then a linear functional $\varphi : M \to \mathbb{C}$ is weak operator continuous if and only if it is strong operator continuous.*

7.2 The pre-dual of $\mathcal{B}$

Recall that we had $\mathcal{B}_\sim = \operatorname{span}\{x \mapsto \sum_k \langle x\xi_k, \eta_k \rangle : \xi_k \eta_k \in H\}$ and $\mathcal{B}_* := \overline{\mathcal{B}_\sim}$. This is the same as taking finite rank operators in $T \in \mathcal{B}(H)$ and taking the functionals $x \mapsto \operatorname{tr}(xT)$.

Remark 7.1. $\mathcal{B}_*$ is the space of trace class operators. Suppose $T \in \mathcal{B}(H)$ has finite rank. We have $|T| = (T^*T)^{1/2}$ by functional calculus. Then $\operatorname{tr}(|T|)$ is the **Schatten-von Neumann 1-norm** of the operator.

Theorem 7.2. *$\mathcal{B} = (\mathcal{B}_*)^*$ via the duality pairing $\mathcal{B} \times \mathcal{B}_* \to \mathbb{C}$ given by $\langle x, \varphi \rangle = \varphi(x)$.*

Here is the idea: Since $\mathcal{B}_* \subseteq \mathcal{B}^*$, we can view its elements as linear functionals on $\mathcal{B}$. But then we can view elements of $\mathcal{B}$ as linear functionals on $\mathcal{B}_*$.

Proof. If $x \in \mathcal{B}$, denote $\Phi_x : \mathcal{B}_* \to \mathbb{C}$ by $\Phi_x(\varphi) = \varphi(x)$. Then $|\Phi_x(\varphi)| \leq \|\varphi\| \cdot \|x\|$, soo $\|\Phi_x\| \leq \|x\|$. So the map $\mathcal{B} \to (\mathcal{B}_*)^*$ sending $x \mapsto \Phi_x$ is a contraction. In fact, $\|\Phi_x\| = \|x\|$ because $\|x\| = \sup_{\xi,\eta\in(H)_1} | \langle x\xi, \eta\rangle$.

To show that this is surjective, take $\Phi \in (\mathcal{B}_*)^*$. Then consider the map $H \times H \mapsto \mathbb{C}$ given by $(\xi, \eta) \mapsto \Phi(\omega_{\xi,\eta})$, where $\omega_{\xi,\eta} = \langle \cdot\xi, \eta\rangle$. So by Riesz-representation, there is an $x \in \mathcal{B}$ such that $\Phi(\omega_{\xi,\eta}) = \langle x, \xi, \eta\rangle$. So $\Phi = \Phi_x$. □

Corollary 7.3. *$(\mathcal{B})_1$ is weak operator compact.*

Proof. This is the topology $\sigma(\mathcal{B}, \mathcal{B}_*)$ topology on $(\mathcal{B})_1$. By the Banach-Alaoglu theorem, this is compact. □

Corollary 7.4. *Let $M \subseteq \mathcal{B}$ be a vector subspace which is weak operator closed. Denote $M_* = \{\varphi|_M : \varphi \in \mathcal{B}_*\}$. Then $(M_*)^* = M$ via the duality pairing $\langle x, \varphi\rangle = \varphi(x)$. Thus, any von Neumann algebra is the dual of some space.*

Remark 7.2. Any C^*-algebra with a pre-dual is a von-Neumann algebra.

Here is some notation:

1. If $X \subseteq H$ is a nonempty subset, then we denote $[X]$ to be the norm closure of span X. We may also use this same notation to denote the orthogonal projection of that space (but this will be clear in context).

2. Let $S \subseteq B(H)$ be nonempty. Then we denote $S' = \{x \in \mathcal{B}(H) : xy = yx \,\forall y \in S\}$ to be the **commutant** of S in $\mathcal{B}(H)$.

Proposition 7.1. *S' is strong operator closed, and it is an algebra. If $S = S*$, where $S^* = \{x^* : x \in S\}$, then S' is a *-algebra. In this case, S' is weak operator closed and is hence a von-Neumann algebra.*

Proof. If $x_i \in S'$ and $x_i \xrightarrow{\text{so}} x \in B(H)$, then

$$xy\xi = \lim_i x_i y\xi = \lim_i y\xi_i\xi = y \lim_i x_i\xi = yx\xi,$$

so $x \in S'$. □

Remark 7.3. Physicists view S' as an algebra of symmetries of states in a system.

We will prove the following theorem next time.

Theorem 7.3 (von Neumann's bicommutant theorem, 1929)**.** *Let $M \subseteq B(H)$ be a *-algebra with $1_M = \mathrm{id}_H$. Then M is a von Neumann algebra if and only if $M = (M')'$.*

Remark 7.4. Some people call this von Neumann's density theorem because it says that the weak operator closure of M is $(M')'$.

Theorem 7.4 (Kaplansky, late 50s)**.** *Let M, M_0 be *-algebras. If the strong operator closure of M_0^* equals M, then $(\overline{M_0}^{\mathrm{so}})_1 = (M)_1$.*

8 von Neumann Bicommutant Theorem and Kaplansky's Density Theorem

8.1 von Neumann's bicommutant theorem

If $S \subseteq \mathcal{B}(H)$, we denote $S' = \{x \in \mathcal{B}(H) : xy = yx \, \forall y \in S\}$ to be the commutant of S. Last time, we had the following results.

Proposition 8.1. *S' is always strong-operator closed, S' is an algebra with unit, and if $S = S^*$, S' is a *-algebra.*

In particular, if $S = S^*$, then S' is a von Neumann algebra.

Theorem 8.1 (von Neumann's bicommutant theorem, 1929)**.** *Let $M \subseteq B(H)$ be a *-algebra with $1_M = \mathrm{id}_H$. Then M is a von Neumann algebra if and only if $M = (M')'$.*

Remark 8.1. Some people call this von Neumann's density theorem because it says that the weak operator closure (or equivalently, the strong operator closure) of M is $(M')'$.

Proposition 8.2. *If $A \subseteq B(H)$ is an algebra, then for all $\xi \in H$, $[A\xi]$ is invariant to all $a_0 \in A$.*

Corollary 8.1. *If $A = A^*$ is a *-algebra, then $[A\xi]$ is invariant to both a_0 and a_0^* for all $a_0 \in A$. Thus, $[A\xi]$ is reductive for a_0: $a_0 \cdot [A\xi_0] = [A\xi_0] \cdot a_0$. So $[A\xi] \in A'$.*

Now we can prove the theorem.

Proof. ($\Longrightarrow$): M'' is weak operator closed, so it is a von Neumann algebra.

($\Longleftarrow$): We have $M \subseteq M''$ and M'' is strong operator closed, so $\overline{M}^{\mathrm{so}} \subseteq M''$. We want to show that M is dense in M'': if $x'' \in M''$, then for any $\xi_1, \dots, \xi_m \in H$ and $\varepsilon > 0$, there is an $x \in M$ such that $\|(x - x'')\xi_i\| < \varepsilon$ for all i.

Step 1: Take first the case $n = 1$. Since $x'' \in M''$, $[x'', [M\xi]] = 0$ (where the outer bracket means commutator); that is, $\underline{x''[M\xi](\xi)} = [M\xi]x''(\xi) \in \overline{M\xi}$. On the other hand, the left hand side is $x''(\xi)$. So $x''\xi \in \overline{M\xi}$. So for any $\varepsilon > 0$, there is an $x \in M$ such that $\|x''\xi - x\xi\| < \varepsilon$.

Step 2: For arbitrary n, take $\widetilde{M} \subseteq \mathcal{B}(H^{\oplus n})$, where $\widetilde{M}$ is the collection of block diagonal matrices that look like

$$\begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix}$$

for $x \in M$. Then $\widetilde{M}$ is SO closed in $\mathcal{B}(H^{\oplus n})$, and $\widetilde{M} \supseteq \{(x'_{i,j})_{1\le i,j\le n} : x'_{i,j} \in M'\}$ If $x = (y_{i,j})_{i,j} \in (\widetilde{M})'$ and

$$e_{i,i} = \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix},$$

then $e_{i,i} \in \widetilde{M}'$. So $\widetilde{M}' = \{(x'_{i,j})_{1\le i,j\le n} : x'_{i,j} \in M'\}$. Now let

$$\widetilde{M''} = \left\{ \begin{bmatrix} x'' & & & \\ & x'' & & \\ & & \ddots & \\ & & & x'' \end{bmatrix} : x'' \in M'' \right\} \subseteq \{(x'_{i,j})_{1\le i,j\le n} : x'_{i,j} \in M'\}'.$$

Then $\widetilde{M''} \subseteq (\tilde{M}')' = \widetilde{M}''$. By the first part applied to

$$\widetilde{x''} = \begin{bmatrix} x'' & & & \\ & x'' & & \\ & & \ddots & \\ & & & x'' \end{bmatrix} \in \widetilde{M}'',$$

we have for $(\xi_1, \dots, \xi_n) \in H^n$ that $\widetilde{x''}\xi \in \widetilde{M}\widetilde{\xi}$. So for every $\varepsilon > 0$, there is an $x = \widetilde{x} \in M$ such that $\|(\widetilde{x''} - \widetilde{x})\widetilde{\xi}\| < \varepsilon$. So $(\sum \|(x'' - x)\xi_i\|^2)^{1/2} < \varepsilon$. $\square$

8.2 Kaplansky's density theorem

Theorem 8.2 (Kaplansky, late 50s)**.** *Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $M_0 \subseteq M$ be a SO-dense *-algebra. Then $(\overline{M_0}^{\mathrm{so}})_1 = (M)_1$. Moreso, $(\overline{M_{0,h}}^{\mathrm{so}})_1 = (M_h)_1$ and $(\overline{M_{0,+}}^{\mathrm{so}})_1 = (M_+)_1$.*

Proof. Step 1: First observe that $\overline{M_{0,h}}^{\mathrm{so}} = M_h$. Indeed, $\overline{M_{0,h}}^{\mathrm{wo}} = M_h$ since $x \mapsto x^*$ is WO-continuous. So if $x_i \to x = x^*$, then $\frac{x_i + x_i^*}{2} \to x$. So $\overline{M_{0,h}}^{\mathrm{so}} = \overline{M_{0,h}}^{\mathrm{wo}} = M_h$.

Step 2: Show that $(\overline{M_{0,h}}^{\mathrm{so}})_1 = (M_h)_1$. We can assume $M_0 = \overline{M_0}^{\mathrm{norm}}$. Let $x = x^* \in (M)_1$, so $\mathrm{Spec}(x) \subseteq [-1, 1]$. Take the bijection $f : [-1, 1] \to [-1, 1]$ sending $t \mapsto \frac{2t}{1+t^2}$. Note that given any $b = b^*$, $f(b)$ makes sense and $\|f(b)\| \le 1$. So there is a $y \in (M_h)_1$ such that $x = \frac{2y}{1+y^2}$. We have that there exist (by step 1) $y_i = y_i^* \xrightarrow{\mathrm{so}} y$ in $(M_0)_h$.

We claim that $\frac{2y_i}{1+y_i^2} \xrightarrow{\text{so}} \frac{2y}{1+y^2} = x$. Indeed,

$$\begin{aligned}\left(\frac{2y_i}{1+y_i^2} - \frac{2y}{1+y^2}\right)\xi &= \frac{1}{1+y_i^2}\left(2y_i(1+y^2) - (1+y_i)^2 2y_i\right)\frac{1}{1+y^2}\xi \\ &= \frac{1}{1+y_i^2}\left((2y_i - 2y) + y_i(y - y_i)\right)\frac{2y}{1+y^2}\xi.\end{aligned}$$

The $2y_i - 2y$ part disappears because of the strong operator convergence, and the left term handles the rest. $\square$

We will do the non self-adjoint part of the proof next time.

9 Kaplansky's Theorem and Polar Decomposition

9.1 Kaplansky's theorem, general case

Let's finish the proof of Kaplansky' theorem.

Theorem 9.1 (Kaplansky, late 50s)**.** *Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $M_0 \subseteq M$ be a SO-dense *-algebra. Then $(\overline{M_0}^{\text{so}})_1 = (M)_1$. Moreso, $(\overline{M_{0,h}}^{\text{so}})_1 = (M_h)_1$ and $(\overline{M_{0,+}}^{\text{so}})_1 = (M_+)_1$.*

In other words, if $x \in M$, there exist $x_i \in M_0$ such that $\|x_i\| \leq \|x\|$ and $x_i \xrightarrow{\text{so}} x$. We have shown this in the case where $x = x^* \in (M_1)$. Let's extend it to the non-self-adjoint case.

Proof. If $x \in (M_+)_1$, then there exist $y_i \in (M_{0,h})_1$ such that $y_i \xrightarrow{\text{so}} \sqrt{x}$. But then $y_i^2 \xrightarrow{\text{so}} (\sqrt{x})^2 = x$; this is because

$$(y_i^2 - y^2)\xi = y_i(y_i - y)\xi + (y_i - y)(y\xi).$$

and $y_i \xrightarrow{\text{so}} y$.

To deal with general $x \in (M)_1$, consider the *-algebra of matrices $M_2(M) \subseteq M_2(\mathcal{B}(H)) = \mathcal{B}(H \oplus H)$. This algebra of matrices is SO-closed, so it is a von Neumann algebra. Moreover, $M_2(M_0)$ is SO dense in $M_2(M)$. By the first part, the operator

$$Y = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \in (M_2(M))_1.$$

The norm of Y is 1 because $Y^* = Y$, and Y^*Y is diagonal with norm $\|x\|^2$. So there exist $Y_i \in (M_2(M_0)_h)_1$ such that $Y_i \xrightarrow{\text{so}} Y$. Since $\|Y_i\| \leq 1$, $\|[Y_i]_{1,2}\| \leq 1$. So we get $[Y_i]_{1,2} \xrightarrow{\text{so}} [Y_i]_{1,2} = x$. $\square$

Corollary 9.1. *Let $M \subseteq \mathcal{B}(H)$ be a *-algebra with unit. The following are equivalent:*

1. *M is a von Neumann algebra (i.e. is WO-closed)*
2. *M is ultraweak-closed.*
3. *$(M)_1$ is ultraweak compact.*

Proof. (1) $\implies$ (3): If M is a von Neumann algebra, then $M = (M_*)^*$, so 3 follows by the Banach-Alaoglu theorem.

(3) $\implies$ (1): This follows from Kaplansky's theorem. $\square$

9.2 Polar decomposition

Definition 9.1. If $x \in \mathcal{B}(H)$, the **left support** $\ell(x)$ is the orthogonal projection onto $[xH]$, and the **right support** $r(x)$ is the orthogonal projection onto $(\ker x)^{\perp} = \overline{\operatorname{im} x^*}$.

Proposition 9.1. *The left and right support satisfy the following:*

1. *$\ell(x)$ is the smallest projection $e \in B(H)$ such that $ex = x$.*
2. *$r(x)$ is the smallest projection $f \in B(H)$ such that $xf = x$.*

So if $x = x^*$, then $\ell(x) = r(x)$.

Definition 9.2. If $x \in \mathcal{B}(H)$ is self adjoint, then $s(x) = \ell(x) = r(x)$ is called the **support** of x.

Recall that a **partial isometry** v is an element such that v^*v and vv^* are projections.[11]

Proposition 9.2. *If $v \in \mathcal{B}(H)$ is a partial isometry, then $\ell(v) = vv^*$ and $r(v) = v = v^*v$.*

Theorem 9.2 (Polar decomposition)**.** *Let $x \in B(H)$. There exist a unique $a \in B(H)_+$ and partial isometry $v \in B(H)$ such that $x = va$ and $v^*v = s(a)$.*

Remark 9.1. This is analogous to the fact that if $\alpha \in \mathbb{C}$, we can express $\alpha = \frac{\alpha}{|\alpha|}|\alpha|$.

Proof. Observe that if $x = va$, then $x^*x = av^*va = a^2$. So $a = \sqrt{x^*x}$.

How should we define v? If $\xi \in r(x)(H)$, then $x\xi = va\xi$ with $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \left\langle a^2\xi, \xi\right\rangle = \|a\xi\|^2$. So we define $v(a\xi) := x\xi$ for $a\xi \in s(a)H$ and $v(\eta) := 0$ if $\eta \perp s(a)(H)$. So v is a partial isometry on H.

For uniqueness, we saw that we must have $a = \sqrt{x^*x}$. If, in addition, $v^*v = s(x)$, then $va\xi = x\xi$. So this choice is forced upon us. $\square$

[11] If one of these is a projection, so is the other.

10 Sups and Infs of Self-Adjoint Operators

10.1 Sups and infs of self-adjoint operators

For $x \in B(H)$, we defined the left support $\ell(x) = [xH]$ and the right support $r(x)$ as the projection onto $(\ker x)^{\perp}$. We had that $\ell(x) = r(x^*)$ and $\ell(x^*) = r(x)$. So if $x = x^*$, then we can define $\ell(x) = r(x) =: s(x)$, the support of x. We also had the following:

Proposition 10.1. *The left and right support satisfy the following:*

1. *$\ell(x)$ is the smallest projection $e \in B(H)$ such that $ex = x$.*
2. *$r(x)$ is the smallest projection $f \in B(H)$ such that $xf = x$.*

Definition 10.1. If $\{e_i\}$ is a family of projections in $\mathcal{B}(H)$, we denote by $\bigvee_i e_i$ the orthogonal projection onto $\overline{\operatorname{span}\{\operatorname{im} e_i\}}$. Denote by $\bigwedge_i e_i$ the orthogonal projection onto $\bigcap_i \operatorname{im} e_i$

Proposition 10.2. *$\bigvee_i e_i$ is the smallest projection e in $B(H)$ such taht $e \geq e_i$ for all i. $\bigwedge_i e_i$ is the largest projection e in $\mathcal{B}(M)$ such that $e \leq e_i$ for all i.*

Proposition 10.3. *If $\{x_i\} \subseteq \mathcal{B}(H)_h$ is uniformly bounded ($\sup_i \|x_i\| < \infty$), then there is a unique $x = x^* \in \mathcal{B}(H)$ such that $x \geq x_i$ for all i and such that if $y = y^* \geq x_i$ for all i, then $y \leq x$. Moreover, if $\{x_i\}$ is an increasing net ($i \leq j \implies x_i \leq x_j$), then $x_i \xrightarrow{\text{so}} x$.*

Remark 10.1. This says that there is a least upper bound $\sup_i x_i$ of $\{x_i\}$ in $\mathcal{B}(H)_h$. Similarly, there exists some $\inf_i x_i$.

Proof. We can assume $0 \leq x_i \leq 1$; if $K = \sup_i \|x_i\|$, then $1 \geq \frac{1}{2K}(x_i + K\mathbf{1}) \geq 0$. For $\xi \in H$, denote $F(\xi, \xi) = \sup_i \langle x_i \xi, \xi \rangle$. Then define $F(\xi, \eta)$ by polarization:

$$F(\xi, \eta) = \frac{1}{4} \sum_{i=0}^{3} i^k F(\xi + i^k \eta, \xi + i^k \eta).$$

Then $|F(\xi, \eta)| \leq \|\xi\| \|\eta\|$ means F is bounded. By the Riesz representation theorem, there is a unique $x \in \mathcal{B}(H)$ such that $\|x\| \leq 1$ and $x \geq 0$ such that $F(\xi, eta) = \langle x\xi, \eta \rangle$ for all $\xi, \eta \in H$. So $\langle x, \xi, \xi \rangle = \sup_i \langle x_i \xi, \xi \rangle$.

To get $x_i \xrightarrow{\text{so}} x$, we want $\|(x - x_i)\xi\| \to 0$ for all $\xi \in H$. We have by functional calculus that

$$\|(x - x_i)\xi\|^2 = \|(x - x_i)^{1/2}(x - x_i)^{1/2}\xi\|^2 \leq \underbrace{\|(x - x_i)^{1/2}\|^2}_{\leq \|x\|} \underbrace{\langle (x - x_i)\xi, \xi \rangle}_{\to 0}.$$

So $x_i \xrightarrow{\text{so}} x$. ∎

Proposition 10.4. *If e is an orthogonal projection,* $\operatorname{Spec}(e) \subseteq \{0, 1\}$.

Proof. Since $e = e^*$, $\operatorname{Spec}(e) \subseteq \mathbb{R}$. Since $e^2 = e$, we must have $\operatorname{Spec}(e) \subseteq \{0, 1\}$. $\square$

Proposition 10.5. *Let $\{e_i\}$ be a family of projections. Then*

$$\bigvee_i e_i = \sup_i e_i, \qquad \bigwedge_i e_i = \inf_i e_i.$$

Proposition 10.6. *Let $\{e_i\}$ be a family of projections. Then*

$$\bigvee_i e_i = \bigvee_{\substack{J \subseteq I \\ J \text{ finite}}} e_J, \qquad e_J = s\left(\sum_{i \in J} e_i\right),$$

and as J increases, $e_J \nearrow \bigvee_i e_i$. In particular, if $|I| < \infty$, then $\bigvee_i e_i = s(\sum_{i \in I} e_i)$.

Remark 10.2. This says that $(\mathcal{P}(B(H)), \leq)$, the projections on H with $\leq$, is a complete lattice.

10.2 Consequences in von Neumann algebras

Proposition 10.7. *If M is a C^*-algebra with unit, then any $x \in M$ is a linear combinations of 4 unitary elements in M. In other words, $M = \operatorname{span} U(M)$.*

Proof. We have $x = \operatorname{Re} x + i \operatorname{Im} x$. But if $a = a^* \in (M)_1$, then we can view it as a function using functional calculus. Then we can split it up into the sum of $t \mapsto t + i\sqrt{1 - t^2}$ and $t \mapsto t - i\sqrt{1 - t^2}$, which are unitary because their ranges are subsets of the unit circle. $\square$

If $M = M^*$, then $[M\xi] \in M'$ for all $\xi \in H$. So if M is a von Neumann algebra, then $[M'\xi] \in M'' = M$. So to check that $x \in M$, it is necessary and sufficient to check that $u'x(u')^* = x$ for all $u \in U(M')$.

Corollary 10.1. *Let M be a von Neumann algebra. Then $\ell(x), r(x) \in M$.*

We will prove this next time. Here is a consequence.

Corollary 10.2. *Let M be a von Neumann algebra. If $x \in M$ and $x = va$ is the polar decomposition, then $v, a \in M$.*

Proof. For any $u' \in U(M')$, we have $u'x(u')^* = x$. On the other hand, $x = u'va(u')^* = u'v(u')^*u'a(u')^*$. Then $v_0 = u'v(u')^*$ is a partial isometry and $a_0 = u'a(u')^* \geq 0$. Then $a = (x^*x)^{1/2} \in M$. So we just need to show that $v \in M$. We have that $r(v_0) = u'r(v)(u')^* = u'r(x)(u')^*$. By uniqueness of the polar decomposition of x, $v = v_0 \in M$. $\square$

Corollary 10.3. *Let M be a von Neumann algebra. If $\{x_i\} \subseteq M$ is uniformly bounded and increasing, then $\sup_i x_i \in M$.*

This is because $x_i \uparrow \sup_i x$ in the SO-topology.

Corollary 10.4. *Let M be a von Neumann algebra. Then $\mathcal{P}(M)$, the projections in M form a complete lattice.*

11 Multiplication Operators on L^2

11.1 Multiplication operators on L^2

Let $(X, \mathcal{F}, \mu)$ be a probability space (we usually assume $\mathcal{F}$ to be countably generated). Then X is measurably isomorphic to $[0,1] \oplus \{1,2,\dots\}$, where $[0,1]$ has (scaled) Lebesgue measure and $\{1,2,\dots\}$ has some scaled counting measure on a subset. If X is a compact metric space, we usually take $\mathcal{F}$ to be the σ-algebra of Borel sets. We choose μ to be regular and assume $\operatorname{supp}(x) = X$. In particular, we may take X to be the spectrum of an operator $x \in \mathcal{B}(H)$.

Consider $L^\infty(X,\mu)$ with $\|\cdot\|_\infty$. If we define $f^*(t) = \overline{f(t)}$, then we get a C^*-algebra structure. If $f \in L^\infty$, we get a multiplication operator on $L^2(\mu)$ given by $M_f(g) = fg$. Moreover, $\|fg\|_2 \le \|f\|_\infty \|g\|_2$, so $M_f \in \mathcal{B}(L^2)$ with $\|M_f\| \le \|f\|_\infty$.

Proposition 11.1. $\|M_f\|_= \|f\|_\infty$.

Proof. Let $X_m = \{t \in X : |f(t) \ge \|f\|_\infty - 1/n\}$. Then $\mu(X_m) > 0$. Then

$$\|M_f(\mathbb{1}_{X_m})\|_2 = \int_{X_n} |f|^2 \, d\mu \ge \left(\left(\|f\|_\infty - \frac{1}{m} \right)^2 \mu(X_m) \right)^{1/2} = (\|f\|_\infty - \frac{1}{m}) \|\mathbb{1}_{X_m}\|_2.$$

So if $\xi_n = \|\mathbb{1}_{X_m}\|_2^{-1} \mathbb{1}_{X_m}$, then we get $\|M_f \xi_n\|_2 \ge \|f\|_\infty - 1/m$. $\square$

Corollary 11.1. *$f \mapsto M_f$ is an isometric *-algebra morphism from L^∞ into $\mathcal{B}(H)$.*

Proof. We have that $f \mapsto M_f$ is a *-algebra morphism, and $M_{\overline{f}} = (M_f)^*$. $\square$

Theorem 11.1. *$\mathcal{A} := \{M_f : f \in L^\infty\} \subseteq \mathcal{B}(L^2)$ is a von Neumann algebra (i.e. it is WO-closed). Moreover, $\mathcal{A}' = \mathcal{A}$ (so $\mathcal{A}$ is maximal abelian in $\mathcal{B}(L^2)$).*

Proof. By von Neumann's bicommutant theorem, we need only show that $\mathcal{A}' = \mathcal{A}$. Let $T \in \mathcal{B}(L^2)$, and suppose that $TM_f = M_f T$ for all $f \in L^\infty$. Then let $\varphi := T(1) \in L^2$.

Define $M_\varphi : L^2 \to L^1$ by $M_\varphi(\psi) = \varphi\psi$ (the image is in L^1 by Cauchy-Schwarz). Then $\|M_\varphi\|_{\mathcal{B}(L^2, L^1)} \le \|\varphi\|_2$ by Cauchy-Schwarz. Both T, M_φ are continuous from $L^2 \to L^1$, and they coincide on $L^\infty \subseteq L^2$ because if $f \in L^\infty$,

$$T(f) = TM_f(1) = M_f T(1) = M_f(\varphi) = f\varphi = M_\varphi(f).$$

Since L^∞ is dense in L^2, $T = M_\varphi$ as operators in $\mathcal{B}(L^2, L^1)$. So $M_\varphi(L^2) \subseteq L^2$.

Why is $\varphi \in L^\infty$? Assume that $\varphi \notin \ell^\infty$. Let $X_n = \{t \in X : |\varphi(t)| \ge n\}$, and let $\xi_n = \mu(X_n)^{-1/2} \mathbb{1}_{X_n}$. Then $\|M_\varphi(\xi_n)\|_2 \ge n$. Letting $n \to \infty$ yields a contradiction. $\square$

11.2 Sups of dominating sequences of operators

Lemma 11.1. *Let $x = x^* \in \mathcal{B}(H)_h$ and let $f_n, g_m \geq 0$ be increasing sequences of continuous functions on* $\operatorname{Spec}(x)$ *that are both uniformly bounded. If* $\sup_n f_n(t) \leq \sup_n g_n(t)$ *for all* $t \in \operatorname{Spec}(x)$*, then* $\sup_n f_n(x) \leq \sup_n g_n(x)$*.*

Let $e_t := \mathbb{1}_{(t,\infty)}(x)$. We can then build all bounded measurable functions using these, and this will give us a functional calculus for all Borel measurable functions.

12 Spectral Scales

12.1 Spectral scales

Last time, we stated the following lemma.

Lemma 12.1. *Let $x = x^* \in \mathcal{B}(H)_h$ and let $f_n, g_n \geq 0$ be increasing sequences of continuous functions on* $\operatorname{Spec}(x)$ *that are both uniformly bounded. If $\sup_n f_n(t) \leq \sup_n g_n(t)$ for all $t \in \operatorname{Spec}(x)$, then $\sup_n f_n(x) \leq \sup_n g_n(x)$.*

Proof. We will prove that for any fixed n and $\varepsilon > 0$, there exists an m_n such that $f_n - \varepsilon \leq g_{m_n}$; this will complete the proof because then $\sup_m g_m(x) \geq f_n(x) - \varepsilon$ for all n and $\varepsilon > 0$.

If $t \in \operatorname{Spec}(x)$, then $f_n(t) - \varepsilon < f_n(t) < \sup_n f_n(t) \leq \sup_m g_m(t)$. So there exists m_n such that $g_{m_n}(t) \geq f_n(t) - \varepsilon$. So there is a neighorhoof V_t of t such that $f_n(s) - \varepsilon < g_{m_n}(s)$ for all $s \in V_t$. By the compactness of $\operatorname{Spec}(x)$, there exist $V_{t_1}, \dots, V_{t_k}$ covering $\operatorname{Spec}(x)$ and corresponding $m_{n_1}, \dots, m_{n_k}$. If we let $m_n := \max\{m_{n_j} : 1 \leq j \leq k\}$, then $f_n(s) - \varepsilon < g_{n_m}(s)$ for all $s \in \operatorname{Spec}(x)$. $\square$

Corollary 12.1. *The* ***spectral scales*** $e_{(-\infty,t]}(x) := \sup\{f(x) : f \in C(\operatorname{Spec}(x)), f \leq \mathbb{1}_{(-\infty,t)}\}$ *are well-defined.*

We have that $\mathbb{1}_{(-\infty,t]}(x) = \bigwedge_{s>t} e_s(x)$. If $Y \subseteq \operatorname{Spec}(x)$ is Borel, then $e_Y = \mathbb{1}_Y(x)$ exists and be called **spectral projections**. These are all contained in the von Neumann algebra generated by x.

Proposition 12.1. *Let $e_{[t,\infty)} = 1 - e_t$. Then*

1. $e_{[\alpha,\beta)} = e_{(-\infty,\alpha)} e_{[\beta,\infty)} = e_\alpha(1 - e_\beta)$.
2. $e_{Y_1} e_{Y_2} = e_{Y_1 \cap Y_2}$.
3. $e_{Y_1} \vee e_{Y_2} = e_{Y_1 \cup Y_2}$.
4. $x e_t \leq t e_t$*, and* $x(1 - e_t) \geq t(1 - e_t)$.

Corollary 12.2. *If $t \leq s$, then*

$$t(e_s - e_t) \leq x(e_s - e_t) \leq s(e_s - e_t).$$

Let $m = \inf \operatorname{Spec}(x)$ and $M = \sup \operatorname{Spec}(x)$. Given a partition $m = t_0 < t_1 < \cdots < t_n = M$, we can construct **Riemann-Darboux sums**

$$s(\Delta) = \sum_{i=1}^{n} t - i - 1(e_{t_i} - e_{t_{i-1}}), \qquad S(\Delta) = \sum_{i=1}^{n} t_i(e_{t_i} - e_{t_{i-1}})$$

If the mesh size is $< \varepsilon$, then $\|S(\Delta) - s(\Delta)\| < \varepsilon$. Then we can define the **vector valued Stieltjes integral** which satisfies

$$x = \int_{-\infty}^{\infty} \lambda \, de_\lambda.$$

Remark 12.1. This integral takes values in $\mathcal{B}(H)$. Since $\mathrm{Spec}(x) \subseteq [m, M]$, this is really an integral over a compact set. The convergence is convergence in norm.

Corollary 12.3. *If $x \in (\mathcal{B}(H)_+)_1$, then there exist projections $\{p_n\}_n$ in $\mathcal{A}$, the von Neumann algebra generated by x, such that $x = \sum_{n \geq 1} 2^{-n} p_n$.*

This is called the **dyadic decomposition** of x.

Proof. Define the projections recursively: Start with $p_1 = e_{[1/2,1)}(x)$, so $\|x - \frac{1}{2}p_1\| \leq 1/2$. Then take $p_2 = e_{[1/4,1/2]}(1 - \frac{1}{2}p_1)$ to be this projection applied to the previous result. Continuing like this, we get all the projections. $\square$

So the von Neumann algebra is generated by x is generated by these projections, as well.

12.2 Cyclic and separating vectors

Definition 12.1. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **cyclic vector** of M if $\overline{M\xi} = H$.

Definition 12.2. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **separating vector** of M if when $x \in M$ satisfies $x\xi = 0$, $x = 0$.

Proposition 12.2. *Let $M = \mathcal{A}$ be an abelian von Neumann algebra. Then ξ is separating if and only if it is cyclic.*

13 Cyclic and Separating Vectors, and The Extension of The Gelfand Transform

13.1 Cyclic and separating vectors

Definition 13.1. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **cyclic vector** of M if $\overline{M\xi} = H$.

Definition 13.2. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **separating vector** of M if when $x \in M$ satisfies $x\xi = 0$, $x = 0$.

Proposition 13.1. *Let $M = \mathcal{A}$ be an abelian von Neumann algebra. If ξ is cyclic, it is separating.*

Proof. If ξ is cyclic and $x\xi = 0$, then $\mathcal{A}(x\xi) = 0$. So $x\overline{\mathcal{A}\xi} = 0$. So $x = 0$. $\square$

Definition 13.3. If $\{p_i\}$ are projections in M with $p_i p_j = 0$, then we define $\sum_i p_i := \bigvee_i p_i$.

Lemma 13.1. *If $\mathcal{A} \subseteq \mathcal{B}(H)$ is an abelian von Neumann algebra, then $\mathcal{A}$ has a separating vector.*

Proof. Let $\{\xi_i\}_{i \in I}$ be a maximal family of unit vectors such that $[A\xi_i]$ is mutually orthogonal. Then $\sum_i [A\xi_i] = 1$. To see why, suppose not. Then for $1 - \sum_i [A\xi_i] \neq 0$, take ξ_0 be a unit vector in the range of $1 - \sum_i [A\xi_i]$. Then for any fixed i, $\langle x\xi_0, y\xi_i \rangle = \langle \xi_0, x^* y \xi_i \rangle = 0$.

This implies that $\{\xi_i\}$ is countable, so let $\xi = \sum_{n \geq 1} 2^{-n} \xi_n$. We claim that if $x \in \mathcal{A}$ and $x\xi = 0$, then $x = 0$. Indeed, if $x\xi = 0$, then $[A\xi_n] x\xi = 0$, so $0 = x[A\xi_n](\xi) = 2^{-n}\xi_n$. This shows that $\xi_n = 0$ for all n, so $x[A\xi_n] = 0$ for all n. So $xH = 0$, making $x = 0$. $\square$

Corollary 13.1. *Let H be separable, and let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an abelian von Neumann algebra with $\xi \in H$ separating for $\mathcal{A}$. Let $p = p_{H_0} = [\mathcal{A}\xi]$. Then the map $\mathcal{A} \mapsto \mathcal{B}(H_0)$ given by $x \mapsto xp$ is a 1 to 1 *-algebra morphism which is SO-SO*[12] *continuous (with SO-SO continuous inverse).*

Remark 13.1. We can also say this is WO-WO continuous.

13.2 Extension of the Gelfand transform

Theorem 13.1. *Let $T \in \mathcal{B}(H)$ be a normal operator, let $\mathcal{A}_T = \{T, T^*\}''$ be the von Neumann algebra generated by T. Assume $\mathcal{A}_T$ has a cyclic vector $\xi \in H$ with $\|\xi\| = 1$. Then there exist a positive, regular Borel measure μ on $X = \mathrm{Spec}(T) \subseteq \mathbb{C}$ of support X, a unitary $U : H \to L^2(X, \mu)$, and an isometric *-morphism $\Phi : \mathcal{A}_T \to \mathcal{B}(L^2(X, \mu))$ implemented spactially by U; i.e. $\Phi(x) = UxU^{-1} \in \mathcal{B}(L^2(X, \mu))$. Moreover, Φ has range $\{M_f : f \in L^\infty(X, \mu)\}$, which is maximal abelian in $\mathcal{B}(L^2(X, \mu))$, and, when restricted to*

[12] This doesn't mean that it's only sort of continuous. But I know you had the thought.

the C^*-algebra generated by T, T^*, is the Gelfand transform. In particular, $\Phi(T^n) = M_{z^n}$, $\Phi((T^*)^n) = M_{\overline{z}^n}$. The measure μ is given by

$$\int_X f\, d\mu = \langle f(T)\xi, \xi \rangle .$$

Uniqueness: If μ_1 is a positive, regular Borel measure on $\mathbb{C}$ with $\mathrm{supp}(\mu_1) = \mathrm{Spec}(T)$ and $\Phi_1 : \mathcal{A}_T \to L^\infty(X, \mu_1)$ extends Γ, then $\mu \sim \mu_1$ and $\Phi_1 = \Phi$.

Proof. Read the Douglas textbook for the proof. □

Now if $T \in \mathcal{B}(H)$ is an arbitrary normal operator, what is its **Borel/L^∞ calculus**? Take a separating $\xi \in H$ for $\mathcal{A}_T = \{T, T^*\}''$. Then $\mathcal{A}_T \mapsto \mathcal{A}_T p \in \mathcal{B}([\mathcal{A}_T \xi])$ identifies $(\mathcal{A}_T, \langle \cdot, \xi, \xi \rangle) \to (L^\infty(\mathrm{Spec}(T)s, \mu), \mu)$.

13.3 Projection geometry

Let $\mathcal{P}(M)$ denote the projections in the von Neumann algebra M.

Definition 13.4. If $e, f \in \mathcal{P}(M)$, then $e \sim f$ if there exists a partial isometry $v \in M$ with $\ell(v) = e$ and $r(v) = f$; i.e. $vv^* = e$ and $v^*v = f$.

Theorem 13.2. *If $x \in M$, then $\ell(x) \sim r(x)$.*

Proof. This is by the polar decomposition of x. □

Theorem 13.3 (Paralellogram rule). *If $e, f \in M$, then $(e \vee f - f) \sim (e - e \wedge f)$.*

Proof. Use the fact that $e \vee f - f = \ell(e(1-f))$, and $e - e \wedge f = r(e(1-f))$. □

Theorem 13.4 (Cantor-Bernstein). *If $e \prec f$ and $f \prec e$, then $e \sim f$.*

14 Geometry of Projections

14.1 Geometry of projections in a von Neumann algebra

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $P(M)$ be the projections in M.

Definition 14.1. Projections $e, f \in P(M)$ are **equivalent** ($e \sim f$) if there exists a partial isometry $v \in M$ such that $vv^* = e$ and $v^*v = f$ (i.e. $\ell(v) = e$, $r(v) = f$). We say that e is **dominated** by f ($e \prec f$) if there exists $f_1 \leq f$ such that $e \sim f_1$ (i.e. there is a partial isometry $v \in M$ such that $vv^* = e$ and $v^*v \leq f$.

Proposition 14.1. *For all $x \in M$, $\ell(x) \sim r(x)$.*

Proof. This follows from the polar decomposition of x: $x = v|x|$. Then $s(|x|) = r(x)$, and $|x| \in s(|x|)Ms(|x|)$. □

Theorem 14.1 (Paralellogram law)**.** *If $e, f \in P(M)$, then $(e \vee f - f) \sim (e - e \wedge f)$.*

Proof. The left hand side is $\ell(e(1-f))$, and the right hand side is $r(e(1-f))$. □

Definition 14.2. The **center** of M is $Z(M) = M' \cap M$.

Definition 14.3. Let $x \in M$. The **central support** of x is the smallest projection z in $Z(M)$ such that $zx = x = xz$. We denote this by $z(x)$.

By taking $\bigwedge_{z_i x = x} z_i$, this exists.

Proposition 14.2. $z(x) = [MxH]$.

Proof. Call the right hand side the projection p. Since $zx = x$, $z \geq p$: $z = uzu^*$ where u is unitary, so $z \geq u\ell(x)u^*$ for all unitary u. So $z \geq \bigvee_u u\ell(x)u^* = [MxH]$ because $\mathrm{span}(U(M)) = M$.

But $px = x$, and $p \in Z(M)$ because $M'MxH = MxH$ and $MMxH = MxH$. By the definition of z, $p \geq z(x)$. □

Theorem 14.2. *Let $e, f \in P(M)$. The following are equivalent:*

1. *$eMf \neq 0$.*
2. *there exist a nonzero $e_1 \leq e$ and a nonzero $f_1 \leq f$ with $e_1 \sim f_1$.*
3. *$z(e)z(f) \neq 0$.*

Theorem 14.3 (Comparison theorem)**.** *Let $e, f \in P(M)$. There exists a projection in $Z(M)$ such that $ep \prec fp$ or $e(1-p) \succ f(1-p)$.*

Proof. Exercise.[13] □

[13]:(

Corollary 14.1. *If $Z(M) = \mathbb{C}$, then $e \prec f$ or $e \succ f$.*

Theorem 14.4 (Schröder-Bernstein type theorem)**.** *If $e \succ f$ and $f \succ e$, then $e \sim f$.*

Proof. Exercise.[14] □

14.2 Vold decomposition

Example 14.1. The left shift on $\ell^2(\mathbb{N})$ is an isometry.

Theorem 14.5 (Vold's decomposition theorem)**.** *If $v \in M$ is an isometry (i.e. $v^*v = 1$), then $v = u \oplus v_0$, where there is a projection p with $u^*u = uu^* = p$, $v_0^*v_0 = 1-p$, $v_0v_0^* \leq 1-p$. (So $v_0^n(v_0^*)^n$ is a decreasing sequence of projections decreasing to 0. This decomposition is unique.*

Remark 14.1. If p is as above, $v^n(v^*)^n \searrow p$. In particular, if $p_0 = (1-p) - v_0v_0^*$, then all $v^np_0(v^n)^*$ are mutually orthogonal.

14.3 Factors and finite projections

Definition 14.4. M is a **factor** if $Z(M) = \mathbb{C}$.

Definition 14.5. $e \in P(M)$ is **abelian** if eMe is abelian.

Example 14.2. $e \in B(H)$ is abelian if and only if e is a 1-dimensional projection.

Definition 14.6. $e \in P(M)$ is a **finite projection** if whenever $f \leq e$ and $f \sim e$, $f = e$.

This is like saying that a set E is finite if the only subset of E that it is in bijection with is E itself.

Remark 14.2. This is equivalent to the following: for any partial isometry $v \in eMe$ with $v^*v = e$, we have $vv^* = e$; i.e. any isometry on eMe is a unitary in eMe.

Definition 14.7. $e \in P(M)$ is **properly infinite** if e has no direct summands in M that are finite, i.e. if $p \in P(M) \cap Z(M)$ with pe finite, then $pe = 0$.

Example 14.3. Consider the von Neumann algebra $\mathbb{C}1 \oplus \mathcal{B}(\ell^2(\mathbb{N}))$. Then $e = 1$ is not a finite projection, but it is not properly infinite. If p is the projection onto the $\mathcal{B}(\ell^2(\mathbb{N}))$ part, then p is properly infinite.

Definition 14.8. A von Neumann algebra M is **finite** if 1 is a finite projection (i.e. any isometry is necessarily a unitary). M is **semifinite** if $1_M = \bigvee_i e_i$ with e_i finite.

Example 14.4. $L^\infty(X)$ is finite (and so is any abelian von Neumann algebra).

[14] :(

Example 14.5. $M_n(\mathbb{C}) = \mathcal{B}(\ell^2_n)$ is finite.

Definition 14.9. A von Neumann algebra M is **type I** if $1_M = \bigvee_i e_i$ with e_i abelian.

Example 14.6. $\mathcal{B}(\ell^2(\mathbb{N}))$ is type I.

Next time, we willl discuss type II and type II von Neumann algebras.

15 Geometry of Projections and Classification of von Neumann Algebras

15.1 Closed graph operators

Definition 15.1. A **closed graph operator** is a linear operator $T : D(T) \to H$, where $D(T) \subseteq H$ is a dense subspace, such that the graph of T, $G_T = \{(\xi, T\xi) : \xi \in D(T)\} \subseteq H \times H$, is closed (i.e. whenever $\xi_n \to 0$ and $T\xi_n \to \eta$, then $\eta = 0$).

Example 15.1. Let $\ell^2(\mathbb{N})$ have its usual orthonormal basis ξ_n. Now define $T_0(\sum c_n\xi_n) = \sum nc_n\xi_n$, which is defined on $D(T_0) = H_0$, the space of finite sums. Now consider $\overline{G_{T_0}}$; there exists some T such that $G_T = \overline{G_{T_0}}$. The space of sequences $\sum_{n=1}^{\infty} c_n\xi_n$ with $\sum_{n=1}^{\infty} |nc_n|^2 < \infty$ is $D(T)$.

15.2 More geometry of projections

Recall some definitions from last time:

Definition 15.2. $e \in P(M)$ is **abelian** if eMe is abelian.

Definition 15.3. $e \in P(M)$ is a **finite projection** if whenever $f \leq e$ and $f \sim e$, $f = e$.

Definition 15.4. A von Neumann algebra M is **finite** if 1 is a finite projection (i.e. any isometry is necessarily a unitary).

Definition 15.5. $e \in P(M)$ is **properly infinite** if e has no direct summands in M that are finite, i.e. if $p \in P(M) \cap Z(M)$ with pe finite, then $pe = 0$.

Lemma 15.1. *Let $e \leq f \in P(M)$ be abelian. Then*

1. *$e = z(e)f$.*
2. *If $z(e) \leq z(f)$, then $e \prec f$.*

Remark 15.1. We always have that if $e \prec f$, then $z(e) \leq z(f)$.

Lemma 15.2. *If $e \in P(M)$ contains no abelian projection (i.e. if $f \leq e$ is abelian, $f = 0$), then there exist $e_1, e_2 \in P(M)$ such that $e_1 \sim e_2$, and $e_1 + e_2 = e$.*

Proof. Take maximal (with respect to inclusion) mutually orthonormal sets $\{e_i\}_I, \{f_i\}_I$ under e. We claim that $\sum_{i\in I} e_i + \sum_{i\in I} f_i = e$; if we call this p and $e - p \neq 0$, then $(e-p)M(e-p)$ is not abelian. Then there exists an $e_0' \sim f_0' \neq 0$ that we can add to the orthonormal sets, contradicting maximality. □

Lemma 15.3. *A projection $e \in P(M)$ is properly infinite if and only if $e = \sum_{n=1}^{\infty} e_n$ with $e_n \sim e$.*

Proof. Use Vold's decomposition. Start by building a family $\{f_n\}$ of mutually orthogonal, mutually equivalent operators. If, say, $e = \sum e_n^0$ with $e_n^0 \sim e_m^0$, then split $\mathbb{N} = \bigcup_{m=1}^{\infty} N_m$ with $|N_m| = \infty$. Then define $e_m = \sum_{k \in N_k} e_k^0$. This is equivalent to $\sum_{m \in \mathbb{N}} e_n^0 = e$. $\square$

Definition 15.6. A projection $e \in P(M)$. is of **countable type** if when $\{e_i\}_{i \in I}$ are mutually orthogonal and $\leq e$, then $|I|$ is countable.

Example 15.2. $\mathcal{B}(\ell^2(\mathbb{N}))$ only has projections of countable type, but $\mathcal{B}(\ell^2(\mathbb{R}))$ has projections not of countable type.

Lemma 15.4. *Let $e, f \in P(M)$, let e be of countable type, and let f be properly infinite. If $z(e) \leq z(f)$, then $e \prec f$.*

Proof. Take $\{e_i\}_{i \in I}$ mutually orthogonal, $\leq e$, and such that $e_i \prec f$ for all i; take a maximal such family with respect to inclusion. We claim that $\sum_i e_i = e$. Indeed, if $p := e - \sum_i e_i \neq 0$, then if $pMf \neq 0$, we contradict maximality: taking x such that $pxf \neq 0$, we get that $\ell(pxf) \sim r(pxf)$. If $pMf = 0$, then $z(p) \leq z(f) = 0$. So I is countable; that is, $e = \sum_n e_n$ with $e_n \prec f$ for all n. But then by induction on n, one builds projections $f_n \leq f$ such that f_n are mutually orthogonal and $e_n \sim f_n$. Now use the previous lemma. $\square$

15.3 Classification of von Neumann algebras

Definition 15.7. A von Neumann algebra M is **semifinite** if $1_M = \bigvee_i e_i$ with e_i finite.

Example 15.3. $\mathcal{B}(\ell^2(\mathbb{N}))$ is semifinite.

Definition 15.8. A von Neumann algebra M is **type I** if $1_M = \bigvee_i e_i$ with e_i abelian.

Example 15.4. $\mathcal{B}(\ell^2(I))$ is of type I for any I.

Definition 15.9. A von Neumann algebra M is **type II** if it is semifinite and has no abelian projections.

So far in this course, we have no examples of type II von Neumann algebras.

Definition 15.10. A von Neumann algebra M is **type II** has no finite projections.

We have no examples yet of this, either.

Definition 15.11. A von Neumann algebra M is **type I finite** if it is of type I and finite. M is of **type I infinite** if it is of type I but has no central finite projection.

Example 15.5. $\mathcal{B}(\ell^2(\mathbb{N}))$ is of type 1 finite. Type 1 infinite algebras looks like $\bigoplus_i \mathcal{B}(\ell^2(J_i)) \otimes L^\infty(X_i)$, where $|J_i| = \infty$.

We can state similar definitions for type II algebras. Here is the key lemma:

Lemma 15.5. *Let $\{e_i\} \subseteq P(M)$ be mutually orhogonal with mutually orthogonal central supports $z(e_i)$.*

1. *If all e_i are abelian, then $\sum_i e_i$ is abelian.*
2. *If all e_i are finite, then $\sum_i e_i$ is finite.*

Theorem 15.1. *Let M be a von Neumann algebras. There exist $p_1, p_2, p_3, p_4, p_5 \in P(M) \cap Z(M)$ with $\sum_{i=1}^{5} p_i = 1$ such that Mp_1 is of type I finite, Mp_2 is type I infinite, Mp_3 is type II_1, Mp_4 is type II infinite, and Mp_5 is type III. So if M is a factor then it is either isomorphic to $M_n(\mathbb{C})$ for some n, $\mathcal{B}(\ell^2(I))$, a type I, and type II, or a type III.*

16 Examples of Factors

16.1 Type I factors

Last time, we discussed the classification of von Neumann algebras by type. The proof is an exercise; it consists of taking maximal orthogonal projections of each type, one type at a time, and looking at the rest of the space.[15]

Proposition 16.1. *If $\{x_i\}_i \subseteq (M)_1$ with mutually orthogonal central supports, then $\sum_i x_i$ is a SO-convergent sum. In fact, if $\{\ell(x_i)\}_i$ are mutually orthogonal, $\{r(x_i)\}_i$ are mutually orthogonal, then $\sum_i x_i$ is SO-convergent.*

We have these 5 types of von Neumann algebras, but we are really interested in factors.

Definition 16.1. A **factor** is a von Neumann algebra M with $Z(M) = \mathbb{C}$.

Example 16.1. Type I finite factors are algebras with $M \cong M_{n\times n}(\mathbb{C}) = \mathcal{B}(\ell_n^2)$.

Example 16.2. Type I infinite factors have $M \cong \mathcal{B}(\ell^2(I))$ for some infinite I.

Lemma 16.1. *If M is a type I factor and $e \in M$ is abelian, then $eMe \cong \mathbb{C}e$.*

Proof. Consider e and $1-e$. We must have $e \succ 1-e$ or $e \prec 1-e$, We can't have the former, so $e \prec 1-e$. Now repeat this with $e_2 \leq 1-e$. We can then find a maximal projection like this. □

16.2 Group von Neumann algebras

Let Γ be a discrete group (not necessarily countable), and let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be the left regular representation: $\lambda(g)(\xi_h) = \xi_{gh}$. We can also take the right regular representation $\rho : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ given by $\rho(g)\xi_h = \xi_{hg^{-1}}$. We have that $\operatorname{span}\lambda(\Gamma)$ is a *-algebra, so its weak closure is a von Neumann algebra.

Definition 16.2. We call $L(\Gamma) := \overline{\operatorname{span}\lambda(\Gamma)}^{\mathrm{wk}} = \lambda(\Gamma)''$ the **group von Neumann algebra** of Γ. Similarly, we have $R(\Gamma) = \rho(\Gamma)''$. We have $[\lambda(\Gamma), \rho(\Gamma)] = 0$, so $L(\Gamma), R(\Gamma)] = 0$.

Define $\tau : L(\Gamma) \to \mathbb{C}$ by $\tau(x) = \langle x\xi_e, \xi_e\rangle$. Notice that

$$\tau(\lambda(g)\lambda(h)) = \langle \xi_{gh}, \xi_e\rangle = \delta_{gh,e} = \delta_{hg,e} = \tau(\lambda(h)\lambda(g)).$$

So $\tau(xy) = \tau(yx)$ for all $x, y \in L(\Gamma)$. Also, τ is a state, and it satisfies the **traciality property** $\tau(1) = 1$. τ is faithful ($\tau(x^*x) = 0 \implies x = 0$), since ξ_e is separating for $L(\Gamma)$.

Proposition 16.2. *If a von Neumann algebra M has a faithful trace, then M is finite.*

[15] Professor Popa said this would be an exercise and then proceeded to write out the proof, which follows this skeleton. I got too lazy to copy down the definition of each individual projection.

Proof. If $u^*u = 1$, then $\tau(1 - uu^*) = 1 - \tau(uu^*) = 1 - \tau(u^*u) = 0$. So $uu^* = 1$. $\square$

Corollary 16.1. *$L(\Gamma)$ is finite.*

When is $L(\Gamma)$ a factor?

Theorem 16.1. *$L(\Gamma)$ is a factor if and only if Γ is infinite conjugacy class (i.e. for any $g \neq e$, $\{hgh^{-1} : h \in \Gamma\}$ is infinite).*

Proof. ($\implies$): Assume there exists some $g \neq e$ such that $\{hgh^{-1} : h \in \Gamma\}$ is finite. This is $\{g_1, \ldots, g_n\} \not\ni e$. Let $z = \sum_{i=1}^n \lambda(g_i)$. Then $\lambda(h)z\lambda(h^{-1}) = z$, and $z(\xi_e) = \sum_{i=1}^n \xi_{g_i} \perp \xi_e$. But since ξ_e is separating, $z \in Z(M)$ and is not a scalar.

($\impliedby$): If we have z with $z(\xi_e) = \sum c_g \xi_g \in \ell^2$ with $z \notin \mathbb{C}1$, then there exists some $g_0 \neq e$ with $c_{g_0} \neq 0$. If $u_h z u_h^* = z$, then $c_{g_0} = c_{hg_0h^{-1}}$ for all h. $\square$

17 Group von Neumann Algebras for ICC Groups

17.1 ICC group von Neumann algebras

Last time, we introduced $L(\Gamma)$, the group von Neumann algebra of Γ. This is the weak operator closure of $\operatorname{span} \lambda(\Gamma)$, where λ is the left regular representation. We saw that $L(\Gamma)$ and $R(\Gamma)$ has the faithful, SO-continuous (and hence WO-continuous) trace state $\tau(x) := \langle x\xi_e, \xi_e \rangle$. This implied that $L(\Gamma)$ is a finite von Neumann algebra.

Theorem 17.1. *$L(\Gamma)$ is a factor if and only if Γ is infinite conjugacy class (i.e. for any $g \neq e$, $\{hgh^{-1} : h \in \Gamma\}$ is infinite).*

Proof. ($\implies$): We did this last time.

($\impliedby$): If Γ is ICC but $z \in Z(\Gamma) \setminus \mathbb{C}1$, then $z(\xi_e) = \sum c_g \xi_g \in \ell^2(\Gamma)$ with $c_{g_0} \neq 0$ for some $g_0 \neq e$. Then for any $h, g \in \Gamma$,

$$\langle \lambda(g) z \lambda(g)^*(\xi_e), \xi_h \rangle = \langle z\xi_e, \xi_h \rangle = c_h.$$

On the other hand, $\lambda(g)^*(\xi_e) = \rho(g)(\xi_e)$, which commutes with z and $\lambda(g)$, so

$$\langle \lambda(g) z \lambda(g)^*(\xi_e), \xi_h \rangle = \langle \rho(g)\lambda(g) z\xi_e, \xi_h \rangle = \left\langle z\xi_e, \lambda_{g^{-1}} \xi_{hg} \right\rangle = \left\langle \sum_g c_g \xi_g, \xi_{g^{-1}hg} \right\rangle = c_{g^{-1}hg}.$$

Take $h = g_0$. Then $c_{g_0} = c_{gg_0g^{-1}}$ for all g, which gives infinitely many equal nonzero coefficients. $\square$

Corollary 17.1. *If Γ is ICC, $L(\Gamma)$ is a II_1 factor.*

Example 17.1. Let S_∞ be the group of finite permutations of $\mathbb{N}$. Then S_∞ is ICC. It is also **locally finite**: for any finite $F \subseteq S_\infty$, there is a finite subgroup of S_∞ containing F.

Example 17.2. Let $\mathbb{F}_n$ be the free group on n generators. This is ICC.

Definition 17.1. Given two groups H_0, Γ_0, $\Gamma_0 \circlearrowright H^{\Gamma_0}$ by left multiplication on the coordinates: $g_0(h_g)_{g \in \Gamma_0} = (h_{g_0^{-1}g})_{g \in \Gamma_0}$. The **wreath product** is the semidirect product $H^{\Gamma_0} \rtimes \Gamma_0$.

Example 17.3. When $H = \mathbb{Z}/2\mathbb{Z}$ and Γ_0, the wreath product is called the **lamp lighter group**. You can think of this as an infinite row of lamps, each lit or unlit. This group is ICC.

Example 17.4. More generally, if $H_0, \Gamma_0 \neq \{1\}$ and Γ is infinite, the wreath product is ICC.

17.2 Distinguishing groups by their von Neumann algebras

More detailed description of $L(\Gamma)$.

Different groups can give rise to different group von Neumann algebras.

Theorem 17.2 (M-vN, 1943)**.** *$L(S_\infty) \neq L(\mathbb{F}_2)$.*

However, there is some collapsing that goes on.

Theorem 17.3. *All ICC locally finite Γ give the same $L(\Gamma)$.*

Here is an open question:
Are $L(\mathbb{F}_n)$ isomorphic or not for different n?

17.3 Multiplication operators on $\ell^2(\Gamma)$

Proposition 17.1. *Any $\xi \in \ell^2(\Gamma)$ defines operators $L_\xi, R_\xi : \ell^2(\Gamma) \to \ell^\infty(\Gamma)$ by*

$$L_\xi(\eta) = \xi\eta = \sum_{g,h} c_g b_h \xi_{gh}, \qquad \text{where} \quad \xi = \sum_g c_g \xi_g, \eta = \sum_h b_h \xi_h.$$

Moreover, $\|L_\xi\|_{\mathcal{B}(\ell^2,\ell^\infty)} \leq \|\xi\|_{\ell^2}$.

Proof. This follows by Cauchy-Schwarz:

$$\sup_{g\in\Gamma} |\xi\eta(g)| = \sup_{g\in\Gamma} \left| \sum_{h\in\Gamma} c_h b_{h^{-1}g} \right| \leq \|\xi\|_{\ell^2} \|\eta\|_{\ell^2}.$$

□

Proposition 17.2. *$D(L_\xi) = L_\xi^{-1}(\ell^2) \subseteq \ell^2$ is a vector subspace, closed in L^2. Moreoverl L_ξ on $D(L_\xi)$ is a densely defined, closed operator on $\ell^2(\Gamma)$.*

Proof. We need to show that L_ξ has closed graph. That is, we need to show that if $\eta_i \to 0$ and $L_\xi(\eta_i) \to \eta$ in ℓ^2, then $\eta = 0$. □

We will do this next time.

18 Recap Episode

18.1 New lore: examples of von Neumann algebras

From 1936 to 1943, Fred Murray and von Neumann published 4 papers titled "On Rings of Operators" I-IV. They treated the question: Are there other von Neumann factors M than $\mathcal{B}(\ell^2(I))$?

Why look at factors? Recall that since $Z(M)$ is abelian, $Z(M) \cong L^\infty(X)$ as abelian algebras for some X. If $Z(M) \neq \mathbb{C}$, and $|X|$ is finite, then $M = \bigoplus_{i=1}^n M_i$, where the M_i are factors.

Proposition 18.1. *If M is a finite dimensional C^*-algebra, then $M = \bigoplus_{i=1}^k M_{n_i \times n_i}(\mathbb{C})$.*

In general, the idea is we should have some kind of decomposition $M \cong \int M_x \, d\mu(x)$.

Theorem 18.1 (Murray-von sNeumann, 1936)**.** *If M is a factor, then either*

1. *It is type I_n (so $M \cong M_{n\times n}(\mathbb{C})$).*
2. *It is type I_∞ (so $M \cong \mathcal{B}(\ell^2(\mathbb{N}))$).*
3. *It is II_1 but not type I finite (so it is infinite-dimensional).*
4. *It is type II_∞ (so it is semifinite)*
5. *It is type III (so it has no finite projections).*

This coincided with the beginnings of ergodic theory:

Theorem 18.2 (von Neumann ergodic theorem, 1932)**.** *Let Γ be a group, and let $\Gamma \circlearrowright X$ be a measure-preserving ergodic action . Then*

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} T^n f - \int f \, d\mu \cdot 1 \right\|_2 \xrightarrow{N\to\infty} 0.$$

Many examples of II_1 factors come from these considerations of ergodic theory.

18.2 Group von Neumann algebras

Last time, we talked about group von Neumann algebras.[16]

Let Γ be a group, and let $L(\Gamma) := \overline{\operatorname{span} \lambda(\Gamma)}^{\mathrm{wk}} = \lambda(\Gamma)'' \subseteq \mathcal{B}(\ell^2(\Gamma))$ be the group von Neumann algebra. We saw that $L(\Gamma)$ has a trace, which implies that $L(\Gamma)$ is finite. If Γ is infinite, then $L(\Gamma)$ is infinite dimensional. So to get II_1 factors, we only need a bit more. We continued this consideration by showing the following:

[16] I don't know why this lecture is recap. But now have enough budget for the rest of the season!

Theorem 18.3. *$L(\Gamma)$ is a II_1 factor if and only if Γ is ICC.*

Example 18.1. S_∞, the finite permutations of $\mathbb{N}$ is ICC.

Example 18.2. $\mathbb{F}_n$, the free group on n generators (with $n \geq 2$), is ICC.

Our proof for this theorem used intuition from Fourier analysis, which we can view as the study of $L(\mathbb{Z})$. For $\xi \in \ell^2(\Gamma)$, we considered $L_\xi : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $L_\xi(\eta) = \xi \cdot \eta$ and saw that $\|L_\xi\|_{\mathcal{B}(\ell^2,\ell^\infty)} \leq \|\xi\|_2$. So $(L_\xi, D(L_\xi))$ is a closed graph operator densely defined on $\ell^2(\Gamma)$. So if $z \in Z$, then $z(\xi_e) = \sum c_g \xi_g$. If $u_h z u_{h^{-1}} = z$ for all h, so $c_g = c_{hgh^{-1}}$ for all g, h. So z is a multiple of the identity.

19 Convolvers in $\ell^2(\Gamma)$

19.1 The group von Neumann algebra of $\mathbb{Z}$

Let's give a more concrete description of the elements of $L(\Gamma)$.

Example 19.1. If $\Gamma = \mathbb{Z}$, then $L(\Gamma) \cong L^\infty(\mathbb{T})$ via the Fourier transform. More precisely, $L^\infty(\mathbb{T}) \cong \{\sum c_n z^n \in \ell^2(\mathbb{Z}) : f * g \in \ell^2(\mathbb{Z})\, \forall g \in \ell^2(\mathbb{Z})\}$ via the map $L^\infty(\mathbb{T}) \to L(\mathbb{Z})$ given by $f \mapsto \sum c_n z^n$, where $c_n = \frac{1}{2\pi}\int f e^{-int}\, d\mu$.

It turns out the general picture looks similar to this case.

19.2 Convolver elements in $\ell^2(\Gamma)$

For $\xi \in \ell^2(\Gamma)$, we get $L_\xi : \ell^2 \to \ell^\infty$, where $L_\xi(\eta) = \xi \cdot \eta$. Then $\|L_\xi\|_{\mathcal{B}(\ell^2,\ell^\infty)} \leq \|\xi\|_{\ell^2}$. We also defined $(L_\xi, D(L_\xi))$ as a closed graph operator on ℓ^2, where $D(L_\xi) = L_\xi^{-1}(\ell^2) = \{\eta \in \ell^2 : \xi \cdot \eta \in \ell^2\}$. This domain contains $\mathbb{C}\Gamma$, the finitely supported series, and the operator has closed graph.

Lemma 19.1. *$L_\xi^* = L_{\xi^*}$, where $\xi^*(g) = \overline{\xi(g^{-1})}$.*

Proof. We can show this for monomials, and by linearity, we can show it for all $\eta \in \mathbb{C}\Gamma$. $\square$

Definition 19.1. An element $\xi \in \ell^2(\Gamma)$ is called a **(left) convolver** if $L_\xi(\ell^2) \subseteq \ell^2$ (i.e. $D(L_\xi) = \ell^2(\Gamma)$.

Corollary 19.1. *ξ is a left convolver if and only if ξ^* is a left convolver.*

Proposition 19.1. *If ξ is a convolver, then $L_\xi : \ell^2 \to \ell^2$ is bounded.*

Proof. This follows from the closed graph theorem. $\square$

Lemma 19.2. *If $\xi, \eta, \zeta \in \ell^2(\Gamma)$ and $\xi \cdot \eta, \eta \cdot \eta \in \ell^2$, then $(\xi \cdot \eta) \cdot \zeta = \xi \cdot (\eta \cdot \zeta)$.*

Corollary 19.2. *If ξ, η are convolvers, then $\xi\eta$ is a convolver, and $L_\xi L_\eta = L_{\xi \cdot \eta}$.*

Corollary 19.3. *ξ is a left convolver if and only if ξ^* is a right convolver.*

Proof. $(\xi \cdot \eta)^* = \eta^* \cdot \xi^*$. $\square$

Theorem 19.1. *Let $LC(\Gamma) := \{L_\xi : \xi$ is a convolver$\}$, $RC(\Gamma) := \{R_\xi : \xi$ is a convolver$\}$. Then $LC(\Gamma)$ and $RC(\Gamma)$ are von Neumann algebras. Moreover, $LC(\Gamma) = L(\Gamma) = R(\Gamma)'$, and $RC(\Gamma) = R(\Gamma) = L(\Gamma)'$.*

Remark 19.1. This theorem tends to have limited utility, but it provides great intuition about what $L(\Gamma)$ and $R(\Gamma)$ looks like.

Proof. $LC(\Gamma)$ is SO closed: Let $\{x_i\}$ be left convolvers such that $L_{\xi_i} \xrightarrow{\text{so}} T \in \mathcal{B}(\ell^2)$. Let $|xi = T(\xi_e$. Then $\|\xi_i - \xi\|_{\ell^2} \to 0$ because $\xi_i = L_{\xi_i}(\xi_e) \to T(\xi_e) = \xi$. But also $L_{\xi_i} \to L_\xi$ in $\mathcal{B}(\ell^2, \ell^\infty)$ because $\|L_{\xi_i - \xi}\|_{\mathcal{B}(\ell^2,\ell^\infty)} \leq \|\xi_i - \xi\|_{\ell^2}$. This implies that L_ξ in $\mathcal{B}(\ell^2, \ell^\infty)$. So ξ is a convolver.

We now have that $LC(\Gamma)$ is a SO-closed *-algebra in $\mathcal{B}(\ell^2(\Gamma))$. So it is a von Neumann algebra. We also have $LC(\Gamma) \supseteq \mathbb{C}\Gamma$, the finitely supported convelvers. So $LC(\Gamma) \supseteq L(\Gamma)$; similarly, $RC(\Gamma) \supseteq R(\Gamma)$. Also, we have $LC(\Gamma)$ commutes with $RC(\Gamma)$: $(\xi \cdot \eta) \cdot \zeta = \xi \cdot (\eta \cdot \zeta)$ gives us $R_\eta(L_\xi(\eta)) = L_\xi(R_\xi(\eta))$.

Thus, $L(\Gamma) \subseteq LC(\Gamma) \subseteq RC(\Gamma)'$ and $R(\Gamma) \subseteq RC(\Gamma) \subseteq LC(\Gamma)'$. This implies that $L(\Gamma)' \supseteq LC(\Gamma)' \supseteq RC(\Gamma)$ and $R(\Gamma)' \supseteq RC(\Gamma)' \supseteq LC(\Gamma)$. We claim that $R(\Gamma)' \subseteq LC(\Gamma)$; this will finish the proof.

Let $T \in R(\Gamma)'$ and let $\xi = T(\xi_e)$. Then

$$T(\xi_g) = T(R_{xi_g}(\xi_e) = R_{\xi_g}(T(\xi_e)) = R_{\xi_g}(\xi) = L_\xi(\xi_g).$$

By linearity, $T = L_\xi$ on $\mathbb{C}\Gamma$. These coincide on a dense subset of $\ell^2(\Gamma)$, so $T = L_\xi$. $\square$

Now we will switch our notation. We will denote $L(\Gamma) = \{\sum c_g n_g : \text{square summable}\}$ endowed with the formal product of series. This is to make the connection with Fourier series more apparent. What does the trace state look like with this notation?

$$\tau\left(\sum c_g u_g\right) = c_e.$$

Notice that

$$\langle x, y \rangle = \tau(y^* x) = \langle x, y \rangle_{\ell^2(\Gamma)}.$$

If we let $M = L(\Gamma)$ with this inner product, then $\ell^2(\Gamma) = \overline{M}^{\|\cdot\|_\tau}$ by the GNS construction.

Next time, we will prove the following theorem in two different ways.

Theorem 19.2. *$L(\mathbb{F}_n) \ncong L(S_\infty)$ for $n \geq 2$.*

20 Distinguishing Group von Neumann Algebras

20.1 $L(\mathbb{F}_2)$ and $L(S_\infty)$ are nonisomorphic

We showed that if Γ is an ICC group, then $L(\Gamma)$ is a II_1 factor. We have many examples of ICC groups.

Example 20.1. S_∞, the group of finite permutations of $\mathbb{N}$ is ICC.

Example 20.2. $\mathbb{F}_n$, the free group on $n \geq 2$ elements, is ICC.

Example 20.3. If $H \neq 1$ and Γ_0 is an infinite group, the wreath product of H and Γ_0 is ICC.

It is not clear that different groups gives different II_1 factors. After all, there is only 1 kind of type I_∞ factor, $\mathcal{B}(\ell^2(\mathbb{N}))$.

Recall that if a factor M has a trace state, then M is a finite factor. Later, we will show that this is an iff.

Definition 20.1. A II_1 factor M (with a trace state τ) has **property Gamma** if for all $x_1, \dots, x_n \in M$ and $\varepsilon > 0$, there exists some $u \in U(M)$ such that $\tau(u) = 0$ and $\|ux_iu^* - x_i\|_\tau < \varepsilon$ for all i.

Here, the norm is $\|x\|_\tau = \tau(x^*x)^{1/2}$. This comes from an inner product, so we may call this $\|x\|_2$.

Proposition 20.1. *If Γ Is locally finite and ICC, then $L(\Gamma)$ has property Gamma.*

Proof. $L(\Gamma) = \{\sum c_g u_g : c_g \in \mathbb{C}, \ell^2 \text{ summable convolvers}\}$. Then $\mathbb{C}\Gamma$ is a *-subalgebra. If $x_1^0, \dots, x_n^0 \in \mathbb{C}\Gamma$, then take a finite subgroup containing them. Now we can pick a unitary convolver outside of this finite subgroup. $\square$

Proposition 20.2. $L(\mathbb{F}_2)$ *does not have property Gamma.*

To prove this, we will prove a lemma.

Lemma 20.1. *Let Γ be an ICC group. Assume exists a set $S \subseteq \Gamma$ and $g_1, g_2, g_3 \in \Gamma$ such that*

1. $S \cup g_1 S g_1^{-1} \cup \{e\} = \Gamma$,
2. S, $g_2 S g_2^{-1}, g_3 S g_3^{-1}$ *are disjoint.*

Then $L(\Gamma)$ does not have property Γ.

Proof. Assume $L(\Gamma)$ has property Gamma. So for any $\varepsilon > 0$, there is a $u \in U(L(\Gamma))$ with $\tau(u) = 0$, $u = \sum c_g u_g$, $c_e = 0$, and $\|u u_{g_i} u^* - u_{g_i}\|_2 < \varepsilon$ for $i = 1, 2, 3$. This says that $\sum_{g \in \Gamma} |c_{g_i g g_i^{-1}} - c_g|^2 < \varepsilon^2$ for $i = 1, 2, 3$. For any $F \subseteq \Gamma$, denote $\nu(F) = \sum_{g \in G} |c_g|^2$ (so $\nu(\Gamma) = 1$). Then, by the triangle inequality in $\| \cdot \|_{\ell^2(S)}$,

$$\left| \left(\sum_{g \in S} |c_g|^2 \right)^{1/2} - \left(\sum_{g \in S} |c_{g_i g g_i^{-1}}|^2 \right)^{1/2} \right| \leq \left(\sum_{g \in S} |c_g - c_{g_i g g_i^{-1}}|^2 \right)^{1/2} < \varepsilon.$$

That is, $|\nu(S)^{1/2} - \nu(g_i S g_i^{-1})^{1/2}| < \varepsilon$. So

$$|\nu(S) - \nu(g_i S g_i^{-1})| \leq 2\varepsilon.$$

But by property (1),

$$\begin{aligned} \nu(\Gamma) &\leq \nu(S) + \nu(g_1 S g_1^{-1}) + \cancel{\nu(\{e\})} \\ &\leq \nu(S) + \nu(S) + 2\varepsilon \\ &= 2\nu(S) + 2\varepsilon \end{aligned}$$

By property (2), we have

$$1 \geq \nu(S) + \nu(g_2 S g_2^{-1}) + \nu(g_3 S g_3^{-1}) \geq 3\nu(S) - 4\varepsilon.$$

This is a contradiction. ☐

Now we can prove the proposition.

Proof. Let $\Gamma = \mathbb{F}_2$ with S, the set of words that start with a^n for $n \neq 0$. Then take $g_1 = a$, $g_2 = b$, and $g_3 = b^{-1}$. These satisfy properties (1) and (2), so by the lemma, $\mathbb{F}_2$ does not have property Gamma. ☐

Remark 20.1. This kind of partition of a group is generally called a **paradoxical partition**. This is a similar kind of thing as what happens in the Banach-Tarski paradox. In that case, $\mathrm{SO}(3) \supseteq \mathbb{F}_2$, and we use this paradoxical partition in that proof.

Corollary 20.1. $L(\mathbb{F}_2) \neq L(S_\infty)$.

20.2 Loss of information from forming $L(\Gamma)$ from Γ

However, this proof is very ad-hoc. It is difficult to tell apart the structure of $L(\Gamma)$ for different groups Γ. The functor $\Gamma \mapsto \mathbb{C}\Gamma$ loses some information. But then $\Gamma \mapsto \overline{\mathbb{C}\Gamma} = L(\Gamma)$ loses a lot of information!

Proposition 20.3. *$\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ and $\mathbb{C}\mathbb{Z}_4$ are both isomorphic to $\mathbb{C}^4$.*

This is because of the torsion. In fact, we have the following fact:

Proposition 20.4. *Let Γ be abelian and countably infinite. Then there is a *-algebra isomorphism* $(L(\Gamma), \tau) \cong (L^\infty([0,1]), \int \cdot\, dm)$

This loss of information happens when going from $\mathbb{C}\Gamma \mapsto L(\Gamma)$.

Proposition 20.5. $\mathbb{C}\mathbb{Z}^n$ *are nonisomorphic for different* n*.*

Proof. The invertible elements in $\mathbb{C}\mathbb{Z}^n$ are $\mathbb{Z}^n(\mathbb{C} \setminus \{0\})$. $\square$

Here is a conjecture:

Theorem 20.1 (Kaplansky)**.** *If* Γ *is torsion free, then* $\mathrm{Inv} = \Gamma \cdot (\mathbb{C} \setminus \{0\})$*.*

This is true if Γ is an **orderable** group. In fact, $\mathbb{F}_n$ is orderable, and many amenable groups are orderable.

Definition 20.2. If Γ is a group, its **group C^*-algebra** is $C^*_r(\Gamma) := \overline{\mathbb{C}(\Gamma)}^{\text{norm}} = \operatorname{span} \overline{\lambda(\Gamma)}^{\text{norm}}$.

$C^*(\Gamma)$ has lots and lots of unitary elements.

Proposition 20.6. *Suppose* Γ *is abelian and torsion-free. If* U_0 *is the connected component of 1,* $U(C^*_r)/U_0 \cong \Gamma$*.*

So this algebra does remember the group.

20.3 Amenable groups

The real property we care about here is amenability. Here is a definition due to von Neumann in the 30s:

Definition 20.3. A group Γ is **amenable** if it has an **invariant mean**, i.e. a state φ on $\ell^\infty(\Gamma)$ such that $\varphi(g^{-1}f) = \varphi(f)$ for all $f \in \ell^\infty$ and $g \in \Gamma$ ($\Gamma \circlearrowright \ell^\infty(\Gamma)$ by left translation on coordinates).

Example 20.4. $\mathbb{Z}^n$ is amenable for any n.

Example 20.5. S_∞ is amenable.

Definition 20.4. Γ has **Følner's property** if for any nonempty, finite $F \subseteq \Gamma$ and $\varepsilon > 0$, there exists a finite $K \subseteq \Gamma$ such that

$$\frac{|FK \triangle K|}{|K|} < \varepsilon.$$

This is same as saying that

$$\frac{|gK \triangle K|}{|K|} < \varepsilon \qquad \forall g \in F.$$

Theorem 20.2. *The Følner property implies ammenability.*

Proof. If Γ has F and is countable, then there exists a sequence $K_n \subseteq \Gamma$ with

$$\frac{|g_i K_n \triangle K_n|}{|K_n|} \xrightarrow{n\to\infty} 0.$$

Choose a non-principal ultrafilter ω on $\mathbb{N}$, and define $\varphi(f) = \lim_{n\to\omega} \frac{1}{|K_n|} \sum_{g\in K_n} f(g)$. This is called a **Banach limit**. So $f \mapsto \varphi(f)$ is linear from $\ell^\infty(\Gamma) \to \mathbb{C}$, $\varphi(1) = 1$, and $\varphi(g_i^{-1} f) = \varphi(f)$ for all i. $\square$

Remark 20.2. We only need to show that $\varphi(g_i^{-1} f) = \varphi(f)$ for the generators of the group.

Example 20.6. $\mathbb{Z}$ is amenable because the sets $K_n = [-n, n]$ gives it the Følner property.

Example 20.7. Locally finite groups are amenable because they satisfy the Følner property.

Proposition 20.7. *If a collection of groups H_i is amenable, then $\bigoplus_i H_i$ is amenable.*

Example 20.8. $\mathbb{Z} \rtimes \mathbb{Z}^n$ and $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}^n$ are ICC and amenable.

Theorem 20.3 (Murray-von Neumann, 1943)**.** *All locally finite ICC groups give the same II_1 factor. In fact, all AFD factors are isomorphic to $L(S_\infty)$.*

Definition 20.5. A II_1 factor M with a trace τ is called **approximately finite dimensional (AFD)** if given any $x_1, \ldots, x_n$ and $\varepsilon > 0$, there exists a finite dimensional von Neumann algebra $B \subseteq M$ and $y_1, \ldots, y_n \in B$ such that $\|x_i - y_i\| < \varepsilon$ for all i

Proposition 20.8. *If Γ is locally finite, then $L(\Gamma)$ is AFD.*

We also have the following remarkable theorem:

Definition 20.6. A II_1 factor is **amenable** if it has an invariant mean (or a **hypertrace**).

Theorem 20.4 (Connes, 1976)**.** *All II_1 factors M that are amenable are isomorphic to $L(S_\infty)$.*

Proposition 20.9. *$L(\Gamma)$ is amenable if and only if Γ is amenable.*

Proposition 20.10. *$\mathbb{F}_2$ is not amenable.*

This gives another proof that S_∞ and $\mathbb{F}_2$ have different group von Neumann algebras.

Corollary 20.2. *$L(\mathbb{F}_2) \ncong L(S_\infty)$.*

21 Amenable Groups and Algebras

21.1 Equivalence of amenability for groups and algebras

Definition 21.1. A II_1 factor M (with a trace state τ) has **property Gamma** if for all $x_1, \ldots, x_n \in M$ and $\varepsilon > 0$, there exists some $u \in U(M)$ such that $\tau(u) = 0$ and $\|ux_iu^* - x_i\|_\tau < \varepsilon$ for all i.

Here $\|x\|_\tau = \tau(x^*x)^{1/2}$. Last time, we showed the folowing:

Theorem 21.1. *1. $L(S_\infty)$ has property Gamma.*

2. For $n \geq 2$, $L(\mathbb{F}_2)$ does not.

Corollary 21.1. *$L(S_\infty) \ncong L(\mathbb{F}_n)$.*

Definition 21.2. Γ is **amenable** if it has an **invariant mean** (i.e. a state $\varphi \in S(\ell^\infty(\Gamma))$ such that $\varphi(gf) = \varphi(f)$ for all $f \ in \ell^\infty$ and $g \in \Gamma$.

Definition 21.3. Γ satisfies Følner's condition if for all nonempty, finite $F \subseteq \Gamma$, for every $\varepsilon > 0$, ther eis a finite $K \subseteq \Gamma$ such that $\frac{|FK \triangle K|}{|K|} < \varepsilon$.

Theorem 21.2. *Γ satisfies Følner's condition if and only if it has an invariant mean.*

We only did the ($\implies$) direction, but we will do the other direction later.

Definition 21.4. A II_1 factor (M, τ) is **amenable** if there exists a state $\varphi \in S(\mathcal{B}(L^2(M,\tau)))$ satisfying $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and $T \in \mathcal{B}(L^2(M))$ (here, $L^2(M) := \overline{M}^{\|\cdot\|_\tau}$). This is equivalent to $\varphi(uTu^*) = \varphi(T)$ for all $u \in U(M)$ and $T \in \mathcal{B}(L^2(M))$. Such a φ is called a **hypertrace**, as $\varphi|_M = \tau$.

In the case where $M = L(\Gamma)$, this is $\varphi \in S(\mathcal{B}(\ell^2(\Gamma)))$.

Theorem 21.3. *Let Γ be an ICC group. $M = L(\Gamma)$ is amenable if and only if Γ is amenable.*

Proof. ($\impliedby$): Take the trace τ on M and extend it to a state on $\mathcal{B}(L^2(M)) = \mathcal{B}(\ell^2(\Gamma))$: take $\widetilde{\tau}(T) = \langle T\xi_e, \xi_e \rangle$ for all $T \in \mathcal{B}(\ell^2(\Gamma))$. Let ψ be an invariant mean on Γ. Define

$$\varphi(T) = \int_\Gamma \widetilde{\tau}(u_g T u_g^*)\, d\psi.$$

By this integration, we mean $\psi((\widetilde{\tau}(u_g T u_g^*))_g)$. Then $\varphi(T) = \varphi(u_h T u_h^*$ for all $h \in \Gamma$, φ is linear, and $\varphi(1) = 1$. So φ is a state on $\mathcal{B}(L^2)$. This says that $\varphi(x_0 T) = \varphi(T x_0)$ for all $T \in \mathcal{B}$ for all $x_0 \in \mathbb{C}\Gamma$.

We want to extend this property to all $\mathcal{B}(L^2)$. Notice also that $\varphi|_M = \tau$. If $x \in (L\Gamma)_1 = (M)_1$ is arbitrary, then let $x_n \in (\mathbb{C}\Gamma)_1$ such that $\|x - x_n\|_2 \to 0$ (by Kaplansky's density theorem). So

$$\begin{aligned}\varphi(xT) &= \varphi((x - x_n)T) + \varphi(x_n T) \\ &= \varphi((x - x_n)T) + \varphi(T x_n) \\ &= \varphi((x - x_n)T) + \varphi(Tx) + \varphi(T(x_n - x))\end{aligned}$$

By Cauchy-Schwarz, we have

$$\begin{aligned}|\varphi((x_n - x)T)| &\leq \varphi((x_n - x)(x_n - x)^*)^{1/2}\varphi(T^*T)^{1/2} \\ &= \tau((x_n - x)(x_n - x)^*)^{1/2}\varphi(T^*T)^{1/2} \\ &= \|x - x_n\|_2 \varphi(T^*T)^{1/2} \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

We get a similar bound for $\varphi(T(x_n - x))$.

($\Longrightarrow$): If $L(\Gamma)$ is amenable and $\varphi \in S(\mathcal{B}(\ell^2(\Gamma)))$ is a hypertrace, then there is an embedding $\ell^\infty(\Gamma) \to \mathcal{B}(\ell^2(\Gamma))$ by $f \mapsto M_f$ on $\ell^2(\Gamma)$. Note that $u_g := \lambda(g)$ satisfies $u_g M_f u_g^* = M_{gf}$. So $\varphi(M_{gf}) = \varphi(u_g M_f u_g^*) = \varphi(M_f)$. In other words, if we define $f \mapsto M_f \mapsto \varphi(M_f)$, we get an invariant mean on Γ. $\square$

Remark 21.1. This is not the proof Murray and von Neumann gave to show that $L(S_\infty)$ and $L(\mathbb{F}_n)$ are non-isomorphic. And von Neumann was the one who formulated the definition of amenable groups!

21.2 Amenability and nonisomorphism of S_∞, $\mathbb{F}_2$, and $S_\infty \times \mathbb{F}_2$

Proposition 21.1. *$\mathbb{F}_2$ is not amenable.*

Proof. Assume φ is an invariant mean on $\ell^\infty(\mathbb{F}_2)$ (where $\mathbb{F}_2$ is the words in letters a and b). Take $A \subseteq \Gamma$ to be all words in a and b that start by a^n with $n \neq 0$. Then $\Gamma = S \cup aS \cup \{e\}$. This gives $1 = \varphi(\mathbb{1}_\Gamma) \leq 2\varphi(\mathbb{1}_S)$ On the other hand, $S, bS, b^{-1}S$ are disjoint. This gives that $3\varphi(\mathbb{1}_S) \leq \varphi(\mathbb{1}_\Gamma) = 1$. This is a contradiction. $\square$

Proposition 21.2. *S_∞ is amenable.*

Proof. It satisfies the Følner property. $\square$

Corollary 21.2. *$L(\mathbb{F}_n) \ncong L(S_\infty)$.*

Proof. The former is not amenable, while the latter is amenable. $\square$

Proposition 21.3. *If $\Gamma_0 \subseteq \Gamma$ and Γ is amenable, then Γ_0 is amenable.*

Proof. We have an embedding $\ell^\infty(\Gamma_0) \to \ell^\infty(\Gamma)$ as follows: take representatives g_n of the cosets in Γ/Γ_0. Then if $f \in \ell^\infty(\Gamma_0)$, we get $\widetilde{f} \in \ell^\infty(\Gamma)$ given by $\widetilde{f}(hg_n) = f(h)$ for all n. This is like reproducing $\ell^\infty(\Gamma_0)$ in $\ell^\infty(\Gamma)$ $|\Gamma/\Gamma_0|$-many times.

If $\varphi \in S(\ell^\infty(\Gamma))$ is an invariant mean, then $\varphi|_{\ell^\infty(\Gamma_0)}$ is an invariant for mean for Γ_0. $\square$

Corollary 21.3. $\mathbb{F}_2 \times S_\infty$ *is not amenable. Moreover, if* $\Gamma \supseteq \mathbb{F}_2$*, then* Γ *is not amenable.*

Corollary 21.4. $L(S_\infty)$*,* $L(\mathbb{F}_2)$*, and* $L(S_\infty \times \mathbb{F}_2)$ *are nonisomorphic.*

Proof. $L(S_\infty)$ is amenable, and the other two are not. $L(\mathbb{F}_2)$ does not have property Gamma, but $L(S_\infty \times \mathbb{F}_2)$ does have property Gamma. $\square$

22 The Hyperfinite II_1 Factor

22.1 Construction

Here is another example of a II_1 factor.

Consider the algebra $R^0 = \bigotimes_{n=1}^{\infty}(M_2(\mathbb{C}), \operatorname{tr})_n$ (this is just algebraic). This is

$$M_2 \xrightarrow{x \mapsto x \otimes 1} M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \xrightarrow{x \mapsto x \otimes 1} M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \longrightarrow \cdots$$

and we take the limit. These inclusions look like

$$x \mapsto \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}.$$

The elements of an infinite tensor product are elements that look like

$$x_1 \otimes x_2 \otimes x_3 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \otimes \cdots$$

for some n.

The algebra R^0 has a trace state given by

$$\tau(x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots) = \operatorname{tr}(x_1) \operatorname{tr}(x_2) \cdots \operatorname{tr}(x_n).$$

This is consistent if we take tr on $M_2(\mathbb{C})$ to be normalized. So $\operatorname{tr}_{M_2(\mathbb{C})}(x) = \operatorname{tr}_{M_{2^{n+1}}(\mathbb{C})}(x)$ via the above inclusions.

R^0 is a *-algebra with the operator norm $\|x\| = \|x\|_{M_{2^n}(\mathbb{C})}$ if $x = x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots$. This is consistent with the inclusions because the operator norm satisfies $\|x \oplus y\| = \max\{\|x\|, \|y\|\}$. This norm satisfies the C^* axiom: $\|x^* x\| = \|x\|^2$. Thus, $(R_0, \|\cdot\|) := \overline{(R^0, \|\cdot\|)}^{\|\cdot\|}$ is a C^*-algebra, and τ extends to a trace state on R_0 (exercise).

Proposition 22.1. *Let $x \in R^0$. If $x \in M_{2^n}(\mathbb{C})$, then*

$$\tau(x) = \int_{U(M_{2^n}(\mathbb{C}))} uxu^* \, du,$$

where the integral is with respect to Haar measure on the unitary group $U(M_{2^n}(\mathbb{C}))$.

Remark 22.1. Since this is in finite dimensions, this is Riemann integral, uniformly convergent in the operator norm

Proof. Call this integral $\Phi(x) \in M_{2^n}(\mathbb{C})$. By the invariance of Haar measure,

$$\Phi(x) = \Phi(u_0 x u_0^*) = u_0 \Phi(x) u_0^*.$$

This implies that $\Phi(x) u_0 = u_0 \Phi(x)$ for all unitary u_0. So $\Phi(x) \in M_{2^n}' \cap M_{2^n} = \mathbb{C}$.

This Φ has the properties of the trace, so by uniqueness of the trace, $\Phi(x) = \tau(x)1$. $\square$

Proposition 22.2. *Let $x \in R^0$. Then $\tau(x^*x) = 0$ if and only if $x = 0$.*

Apply the GNS construction for (R_0, τ) to get the representation $(\pi_\tau, H_\tau, \xi_\tau = \hat{1})$; recall that $H_\tau = \overline{R_0}^{\|\cdot\|_\tau}$. So $x \mapsto \pi_\tau(x)$, which is left-multiplication by x on $\widehat{R_0} = H_\tau^0$, which contains $\widehat{R^0}$ as a dense subset. Also, we have $\tau(x) = \langle x\hat{1}, \hat{1}\rangle_{H_\tau}$.

π_τ is isometric because it is isometric on each $M_{2^n}(\mathbb{C})$ (since it is an injective morphism of C^*-algebras).

Definition 22.1. The **hyperfinite** II_1 **factor** (R, τ) is $\overline{\pi_\tau(R_0)}^{\text{wo}}$, endowed with the trace state $\tau(x) = \langle x\hat{1}, \hat{1}\rangle$.

This is a von Neumann algebra.

22.2 R is a II_1 factor

Proposition 22.3. *τ is faithful on R ($\tau(x^*x) = 0 \iff x = 0$) if and only if ξ_τ is separating for R.*

Proof. ($\implies$): $\tau(x^*x) = \|x\xi_\tau\|_{H_\tau}^2$. We have $R = \overline{\pi_\tau(R^0)}^{\text{wo}}$m but we could have taken right multiplication in the GNS construction. Also λ and ρ, left and right multiplication, commute. So $\rho(y)(x\hat{1} =)$ for all R^0. Thus, $\rho(y)(x\hat{1}) = 0$ for all $y \in R^0$, so $[R, \rho(R^0)] = 0$. So if $x\hat{1} = 0$, then $\rho(y)x(\hat{1}) = 0 = x(\rho(y)\hat{1}) = xy = x(\hat{y}) = 0$. This implies that $x(H_\tau) = 0$, so $x = 0$. $\square$

τ on R is a faithful trace. In particualar, R is finite.

Proposition 22.4. *(R, τ) is a II_1 factor.*

Proof. Assume $z \in (Z(M))_1$. Then there exists some $x_i \in (R_0)_1$ such that $\pi_\tau(x_i) \xrightarrow{\text{so}} z$ by Kaplansky's theorem. We have

$$\|\pi_\tau(x_i)\hat{1} - z(\hat{1})\|_{H_\tau} \to 0 \iff \|x_i - z\|_\tau \to 0,$$

where $\|x_i - z\|_\tau = \|ux_ix^* - z\|_\tau$ for any unitary u. But for each fixed i, if $x \in M_{2^{n_i}}(\mathbb{C})$, then $\|\int ux_iu^* - z\,du\|_\tau \leq \|ux_iu^* - z\|_\tau$ for all unitary u. But the left hand side is $\|\tau(x_i) - z\|_\tau$. Therefore, $\|z - c\hat{1}\|_\tau = 0$. But τ is faithful, so $z = c\hat{1}$ is a scalar. $\square$

23 Every II_1 Factor Has a Trace

This note is based on a set of slides Professor Popa used for the lecture.

23.1 Theorem and the hyperfinite II_1 factor

Last time, we defined the hyperfinite II_1 factor by constructing R_0 with the trace state τ. We can define the Hilbert space $L^2(R_0)$ as the completion of R_0 with respect to the Hilbert-norm $\|y\|_2 = \tau(y^*y)^{1/2}$, and denote $\widehat{R_0}$ as the copy of R_0 as a subspace of $L^2(R_0)$.

For each $x \in R_0$, define the operator $\lambda(x)$ on $L^2(R_0)$ by $\lambda(x)(\widehat{y}) = \widehat{xy}$ for all $y \in R_0$. Note that $x \mapsto \lambda(x)$ is a *-algebra morphism $R_0 \to \mathcal{B}(L^2)$ with $\|\lambda(x)\| = \|x\|$ for all x. Moreover, $\langle \lambda(x)(\hat{1}), \hat{1}\rangle_{L^2} = \tau(x)$.

We can similarly define $\rho(x)$ to be the right multiplication operator. Then λ and ρ commute. Last time, we showed that the von Neumann algebra $R = \overline{\lambda(R_0)}^{\text{wo}}$ is a II_1 factor.

One other way to define R is as the completion of R_0 in the topology of convergence in hte norm $\|x\|_2 = \tau(x^*x)^{1/2}$ of sequences that are bounded in the operator norm. Notice that, in both definitions, τ extends to a trace state on R. If one denotes by $D_0 \subseteq R_0$ the natural "diagonal subalgebra," then $(D_0, \tau|_{D_0})$ coincides with the algebra of dyadic step functions on $[0,1]$ with the Lebesgue integral. So its closure in R in the above topology, $(D, \tau|_D)$ is just $(L^\infty([0,1]), \int d\mu)$.

Also, (R_0, τ) (and thus R) is completely determined by the sequence of partial isometries $v_1 = e^1_{1,2}$, $v_n = (\prod_{i=1}^{n-1} e^i_{2,2})e^n_{1,2}$ for $n \geq 2$ with $p_n = v_n v_n^*$; these satisfy $\tau(p_n) = 2^{-n}$ and $p_n \sim 1 - \sum_{i=1}^n p_i$.

Theorem 23.1. *Let M be a von Neumann factor. The following are equivalent:*

1. *M is a **finite** von Neumann algebra; i.e. if $p \in P(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (any isometry in M is necessarily a unitary element).*
2. *M has a **trace state** (i.e. a functional $\tau : M \to \mathbb{C}$ that is positive, $\tau(x^*x) \geq 0$, $\tau(1) = 1$, and $\tau(xy) = \tau(yx)$ for all $x, y \in M$).*
3. *M has a trace state τ that is **completely additive** (i.e. $\tau(\sum_i p_i) = \sum_i \tau p_i$) for for all mutually orthogonal projections $\mathcal{P}(M)$.*
4. *M has a trace state τ that is **normal** (i.e. $\tau(\sup_i x_i) = \sup_i \tau(x_i)$ if $(x_i)_i \subseteq (M_+)_1$ is an increasing net).*

So a von Neumann factor is finite if and only if it is tracial. Moreover, such a factor has the unique trace state τ, which is automatically normal, faithful, and satisfies $\overline{\text{co}}\{uxu^* : u \in U(M)\} \cap \mathbb{C}1 = \{\tau(x)1\}$ for all $x \in M$.

These are progressively stronger conditions, so we need only show that (4) $\implies$ (1). We need some lemmas.

23.2 Projections in a finite von Neumann factor

Lemma 23.1. *If a von Neumann factor M has a minimal projection, then $M = \mathcal{B}(\ell^2(I))$ for some I. Moreover, if $M = \mathcal{B}(\ell^2(I))$, then the following are equivalent:*

1. *M has a trace*
2. *$|I| < \infty$.*
3. *M is finite, i.e. if $u \in M$ with $u * u = 1$, then $uu^* = 1$.*

Proof. If we have a trace in finite dimensions, split $1 = p_1 + p_2$ into two projections onto infinite dimensional subspaces. Since trace is additive and $p_1 \sim p_2$, $\tau(p_1) = \tau(p_2) = 1$. Do the same with p_2 to get p_3 and p_4. But then $1 = p_1 + p_3 + p_4$, where $\tau(p_1) = \tau(p_3) = \tau(p_4)$ because these projections are equivalent. But this gives $\tau(p_1) = 1/3$, which is a contradiction. □

Lemma 23.2. *If M is finite, then*

1. *If $p, q \in P(M)$ are such that $p \sim q$, then $1 - p \sim 1 - q$.*
2. *pMp is finite for all $p \in P(M)$; i.e. if $q \in P(M)$ and $q \leq p$ with $q \sim p$, then $q = p$.*

Proof. Use the comparison theorem. □

Lemma 23.3. *If M is a von Neumann factor with no atoms (so $p \in P(M)$ has $\dim(pMp) = \infty$), then there exist $P_0, P_1 \in P(M)$ with $P_0 \sim P_1$ and $P_0 + P_1 = p$.*

So we can split p into two equivalent projections.

Proof. Consider the family $\mathcal{F} = \{(p_i^0, p_i^1)_i : p_i^0, p_j^1 \text{ mut. orth.}, \leq p, p_i^0 \sim p_i^1\}$ with the ordering from inclusion. Obtain a maximal element of $\mathcal{F}$. If $(p_i^0, p_i^1)_{i \in I}$ is a maximal element, then $P_0 = \sum_i p_i^0$ and $P_1 = \sum_o p_i^1$ will do; if not then the comparison theorem gives a contadiction. □

Lemma 23.4. *If M is a factor with no minimal projections, there exists a sequence of mutually orthogonal projections $(p_n)_n \subseteq P(M)$ such htat $p_n \sim 1 - \sum_{i=1}^n p_i$ for all n.*

Proof. Apply the previous lemma recursively. □

Lemma 23.5. *If M is a finite factor and $(p_n)_n$ are as in the previous lemma, then*

1. *If $p \prec p_n$ for all n, then $p = 0$. Equivalently, if $p \neq 0$, there exists some n such that $p_n \prec p$. Moreover, if n is the first integer such that $p_n \prec p$ and $p_n' \leq p$ with $p_n' \sim p_n$, then $p - p_n' \prec p_n$.*

2. *If $(q_n)_n \subseteq P(M)$ is increasing, $q_n \leq q \in P(M)$, and $q - q_n \prec p_n$ for all n, then $q_n \nearrow q$ (with SO convergence).*

3. $\sum_n p_n = 1$.

Proof. If $p \sim p'_n \leq p_n$ for all n, then $P = \sum_n p'_n$, and $P_0 = \sum_k p'_{2k+1}$ satisfy $P_0 < P$ and $P_0 \sim P$. This contradicts the finiteness of M. □

Lemma 23.6. *Let M be a finite factor without atoms. If $p \in P(M)$ is nonzero, then there is a unique infinite sequence $1 \leq n_1 < n_2 < \cdots$ such that p decomposes as $p = \sum_{k\geq 1} p'_{n_k}$ for some $(p_{n_k})_k \subseteq P(M)$ with $p'_{n_k} \sim p_{n_k}$ for all k.*

Proof. Apply part (1) of the previous lemma recursively. By part (2), the sum converges to p. □

Definition 23.1. If M is a finite factor without atoms, the **dimension** is $\dim : P(M) \to [0,1]$ given by $\dim(p) = 0$ if $p = 0$ and $\dim(p) = \sum_{k=1}^{\infty} 2^{-n_k}$ if $p \neq 0$, where $n_1 < n_2 < \cdots$ are gven by the previous lemma.

Lemma 23.7. $\dim$ *satisfies the following conditions:*

1. $\dim(p_n) = 2^{-n}$.

2. *If $p, q \in P(M)$, then $p \leq q$ iff $\dim(p) \leq \dim(q)$.*

3. $\dim$ *is completely additive: if $q_i \in P(M)$ are mutually orthogonal, then* $\dim(\sum_i q_i) = \sum_i \dim(q_i)$.

23.3 The Radon-Nikodym trick

We claim that dim extends to the trace τ on $(M)_+$ in the following way. If $0 \leq x \leq 1$, then $x = \sum_{n=1}^{\infty} 2^{-n} e_n$. So if we put $\tau(x) = \sum 2^{-n} \dim(e_n)$, this is well-defined. Now if $x \in (M)_h$, we can take $\tau(x) = \tau(x_+) - \tau(x_-)$. And then we can extend this to M. But we have a problem; we cannot tell that this τ is linear.

Lemma 23.8 ("Radon-Nikodym trick")**.** *Let $\varphi, \psi : P(M) \to [0,1]$ be completely additive functions with $\varphi \neq 0$ and $\varepsilon > 0$. There exists a $p \in P(M)$ with $\dim(p) = 2^{-n}$ for some $n \geq 1$ and $\theta \geq 0$ such that $\theta\varphi(q) \leq \psi(q) \leq (1+\varepsilon)\theta\varphi(q)$ for all $q \in P(pMp)$.*

Intuitively, we want to think of φ, ψ like measures. In other words, we can take a small part of the space where φ and ψ are almost multiples of each other.

Proof. Denote $\mathcal{F} = \{p : \exists n \text{s.t.} p \sim p_n\}$. We may assume φ is faithful: take a maximal family of mutually orthogonal nonzero projections (e_i) with $\varphi(e_i) = 0$ for all i. Then let $f = 1 - \sum_i e_i \neq 0$ (because $\varphi(1) \neq 0$); it follows that φ is faithful on fMf, and by replacing with some $f_0 \le f$ in $\mathcal{F}$, we may also assume $f \in \mathcal{F}$. Thus, proving the lemma for M is equivalent to proving it for fMF, which amounts to assuming φ is faithful.

If $\psi = 0$, then we take $\theta = 0$. If $\psi \neq 0$, then by replacing φ by $\varphi(1)^{-1}\varphi$ and ψ by $\psi(1)^{-1}\psi$, we may assume that $\varphi(1) = \psi(1) = 1$. We claim that this implies: There exists a $f \in \mathcal{F}$ such that for all $g_0 \in \mathcal{F}$ with $g_0 \le g$, we have $\varphi(g_0) \le \psi(g_0)$.

If not, then for all $g \in \mathcal{F}$, there is a $g_0 \in \mathcal{F}$ with $g_0 \le g$ such that $\varphi(g_0) > \psi(g_0)$. Tkae a maximal family of mutually orthogonal projections $(g_i)_i \subseteq \mathcal{F}$ with $\varphi(g_i) > \psi(g_i)$ for all i. If $1 - \sum_i g_i \neq 0$, then take $g \in \mathcal{F}$ with $g \le 1 - \sum_i g_i$ and apply this condition to get $g_0 \in \mathcal{F}$ with $g_0 \le g$ and $\varphi(g_0) > \psi(g_0)$, contradicting maximality. Thus,

$$1 - \varphi\bigg(\sum_i g_i\bigg) = \sum_i \varphi(g_i) > \sum_i \psi(g_i) = \psi\bigg(\sum_i g_i\bigg) = \psi(1) = 1,$$

a contradiction. So this case is impossible.

Define $\theta = \sup\{\theta' : \theta'\varphi(g_0) \le \psi(g_0)\ \forall g_0 \le g, g_0 \in \mathcal{F}\}$. Then $1 \le \theta < \infty$, and $\theta\varphi(g_0) \le \psi(g_0)$ for all $g_0 \in \mathcal{F}$ with $g_0 \le g$. Moreover, by definition of θ, there exists some $g_0 \in \mathcal{F}$ with $g_0 \le g$ such that $\theta\varphi(g_0) > (1+\varepsilon)^{-1}\psi(g_0)$. We now repeat the argument for ψ and $\theta(1+\varepsilon)\varphi$ on g_0Mg_0 to prove the following:

We claim that there exists some $g' \in \mathcal{F}$ with $g' \le g_0$ such that for all $g_0' \in \mathcal{F}$ with $g_0' \le g_0$, we have $\psi(g_0') \le \theta(1+\varepsilon)\varphi(g_0')$. If not, then for all $g' \in \mathcal{F}$ with $g' \le g_0$, there is a $g_0' \le g'$ in $\mathcal{F}$ such that $\psi(g_0') > \theta(1+\varepsilon)\varphi(g_0')$. But then take a maximal family of mutually orthogonal $g_i' \le g_0$ such that $\psi(g_i') \ge \theta(1+\varepsilon)\varphi(g_i')$. Using one of the previous lemmas, we get $\sum_i g_i' = g_0$. Then $\psi(g_0) \ge \theta(1+\varepsilon)\varphi(g_0) > \psi(g_0)$. This is a contradiction. So the claim holds for some $g; \in \mathcal{F}$ with $g' \le g_0$. Taking $p = g'$, we get that any $q \in \mathcal{F}$ under p satisfies both $\theta\varphi(q) \le \psi(q)$ and $\psi(q) \le \theta(1+\varepsilon)\varphi(q)$. By complete additivity of φ and ψ, using a previous lemma, we are done. □

Now apply the lemma to $\psi = \dim$ and φ to be a vector state on $M \subseteq \mathcal{B}(H)$ to get the following:

Lemma 23.9. *For all $\varepsilon > 0$, there exists some $p \in P(M)$ with $\dim(p) = 2^{-n}$ for some $n \ge 1$ and a vector state φ_0 on pMp such that for all $q \in P(pMp)$, $(1+\varepsilon^{-1}\varphi_0(q) \le 2^n \dim(q) \le (1+\varepsilon\varphi_0(q)$.*

We want to reproduce the linearity of the dimension function on pMp to the whole space.

Lemma 23.10. *With p, φ_0 as above, let $v_1 = p, v_2, \dots, v_{2^n} \in M$ such that $v_iv_i^* = p$ and $\sum_i v_iv_i^* = 1$. Let $\varphi(x) := \sum_{i=1}^{2^n} \varphi_0(v_ixv_i^*)$ for $x \in M$. Then φ is a normal state on M satisfying $\varphi(x^*x) \le (1+\varepsilon)\varphi(xx^*)$ for all $x \in M$.*

Proof. Note first that $\varphi_0(x^*x) \le (1+\varepsilon)\varphi_0(xx^*)$ for all $x \in pMp$ (do it first for when x is a partial isometry, then for x with x^*x having finite spectrum). To deduce the inequality for φ itself, note that if $\sum_j v_j^* v_i = 1$, then for any $x \in M$,

$$\begin{aligned}
\varphi(x^*x) &= \sum_i \varphi_0\bigg(v_i x^* \bigg(\sum_j v_j^* v_j\bigg) x v_i^*\bigg) \\
&= \sum_{i,j} \varphi\varphi_0((v_i x^* v_j^*)(x_j x v_i)) \\
&\leq (1+\varepsilon^2) \sum_{i,j} \varphi_0((v_j x v_i)(v_i x^* v_j^*)) \\
&= \cdots \\
&= (1+\varepsilon^2)\varphi(xx^*). \qquad \square
\end{aligned}$$

Lemma 23.11. *If φ is a state on M that satisfies $\varphi(x^*x) \leq (1+\varepsilon)\varphi(xx^*)$ for all $x \in M$, then $(1+\varepsilon)^{-1}\varphi(p) \leq \dim(p)(1+\varepsilon)\varphi(p)$ for all $p \in P(M)$.*

Proof. By complete additivity, it is sufficient to prove it for $p \in \mathcal{F}$, when we have $v_1, \dots, v_{2^n}$ as in the previous lemma. Then $\varphi(p) = \varphi(v_j^* v_j) \leq (1+\varepsilon)\varphi(v_j v_j^*)$ for all j, so

$$2^n \varphi(p) \leq (1+\varepsilon)^2 \sum_j \varphi(v_j v_j^*) = (1+\varepsilon)^2 2^n \dim(p).$$

Similarly, $2^n \dim(p) = 1 \leq (1+\varepsilon)^2 2^n \varphi(p)$. $\square$

23.4 Proof of the theorem

Now we can prove the theorem.

Proof. Define τ as mentioned before. By the previous lemma, for every $\varepsilon > 0$, there is a normal state φ on M such that $|\tau(p) - \varphi(p)| \leq \varepsilon$ for all $p \in P(M)$. By definition of $|tau$ and the linearity of φ, this implies that $|\tau(x) - \varphi(x)| \leq \varepsilon$ for all $x \in (M_+)_1$. So $|\tau(x) - \varphi(x)| \leq 4\varepsilon$ for all $x \in (M)_1$. This implies that $\tau(x+y) - \tau(x) - \tau(y)| \leq 8\varepsilon$ for all $x, y \in (M)_1$. Since $\varepsilon > 0$ was arbitrary, we get that τ is a linear state on M.

By definition of τ, we also have $\tau(uxu^*) = \tau(x)$ for all $x \in M$ and $u \in U(M)$. So τ is a trace state. From the above argument it also follows that norm limit of normal states φ, so τ is normal as well. $\square$

This theorem also has a generalization.

Theorem 23.2. *Let M be a von Neumann algebra that is countably decomposable (i.e. any family of mutually orthogonal projections is countable). The following are equivalent:*

1. *M is a **finite** von Neumann algebra; i.e. if $p \in P(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (so any isometry in M is necessarily a unitary element).*

2. M has a faithful, normal (equivalently completely additive) trace state τ.

*Moreover, if M is finite, then there exists a unique normal faithful **central trace**, i.e. a linear positive map* $\operatorname{ctr} : M \to Z(M)$ *that satisfies* $\operatorname{ctr}(1) = 1$, $\operatorname{ctr}(z_1 x z_2) = z_1 \operatorname{ctr}(x) z_2$, *and* $\operatorname{ctr}(xy) = \operatorname{ctr}(yx)$ *for al l$x, y \in M$ and $z_i \in Z$.*

Any trace τ on M is of the form $\tau = \varphi_0 \circ \operatorname{ctr}$ for some state φ on Z. Also, $\overline{\operatorname{co}}\{uxu^* : u \in U(M)\} \cap Z = \{\operatorname{ctr}(x)\}$ for all $x \in M$.

The central trace should be thought of like a conditional expectation onto $Z(M)$.

24 The Group Measure Space von Neumann Algebra Construction

24.1 Measure-preserving actions of groups

Up to now, our only examples of II_1 factors have been

- the hyperfinite II_1 factor R,
- the group von Neumann algebra $L(\Gamma)$, where Γ is ICC.

Here is another class of examples of II_1 factors.

Definition 24.1. Let Γ be a discrete, countable group, and let (X, μ) be a standard, nonatomic probability space. A **measure-preserving action** $\Gamma \circlearrowright^\sigma X$ is a collection of measure-preserving ($\mu(a\sigma_g^{-1}A) = \mu(A)$) maps $\sigma_g : X \to X$ that are invertible (mod null sets) such that $\sigma : \Gamma \to \mathrm{Aut}(X, \mu)$ sending $g \mapsto \sigma_g$ is a group homomorphism.

By $\mathrm{Aut}(X, \mu)$, we mean automorphisms of X as a measure space. A measure-preserving α gives rise to an map $\alpha^* \in \mathrm{Aut}(L^\infty(X, \mu), \int d\mu)$ given by $\alpha^*(f) = f \circ \alpha^{-1}$. The action $\Gamma \circlearrowright^\sigma (X, \mu)$ induces σ^*, an action of Γ on $(L^\infty(X, \mu), \int d\mu)$ by $(\sigma^*)_g(f) = f \circ (\sigma_g)^{-1}$; in particular, $(\sigma^*)_g(\sigma^*)_h = (\sigma^*)_{gh}$. That is, we gave a homomorphism $\sigma^* : \Gamma \to \mathrm{Aut}(L^\infty(X, \mu), \int \cdot\, d\mu)$, where this is the group of automorphisms of $L^\infty(X)$ preserving $\int \cdot\, d\mu$. We will denote this action by $\Gamma \circlearrowright^\sigma (L^\infty(X, \mu), \int \cdot\, d\mu)$, suppressing the star notation.

Surprisingly, we can go back!

Theorem 24.1 (von Neumann)**.** *Let $\beta \in \mathrm{Aut}(L^\infty(X, \mu), \int \cdot\, d\mu)$. Then there exists a unique $\alpha \in \mathrm{Aut}(X, \mu)$ such that $\alpha^* = \beta$.*

24.2 Construction of the algebra

Here, $A = L^\infty$ is a von Neumann algebra. Form the vector space $A\Gamma$ of finitely supported sums $\sum_g a_g u_g : a_g \in A\}$. We can turn this into an algebra by introducing the multiplication

$$(a_g u_g) \cdot (a_h u_h) = a_g \sigma_g(a_h) u_{gh}.$$

This gives us an algebra where $u_g a_h u_g^{-1} = \sigma_g(a_h)$. Moreover, this is a *-algebra by

$$(a_g u_g)^* = u_{g^{-1}} \overline{a_g} = \sigma_{g^{-1}}(\overline{a_g}) u_{g^{-1}}.$$

We also have the functional

$$\tau\left(\sum a_g u_g\right) = \int a_e \, d\mu.$$

Proposition 24.1. *τ is a trace state.*

Proof. We want to show that $\tau((a_g u_g)\cdot(a_h u_h)) = \tau((a_h u_h)\cdot(a_g u_g))$. That is, we want to show that

$$\delta_{gh,e}\int a_g\sigma_g(a_h)\,d\mu = \delta_{hg,e}\int a_h\sigma_h(a_g)\ ,d\mu.$$

Replacing $h = g^{-1}$, this is

$$\int a_g\sigma_g(a_{g^{-1}})\,d\mu = \int a_h\sigma_h(a_{h^{-1}})\,d\mu$$

If we apply $\sigma_{h^{-1}}$ to the right hand side, since $\sigma_{h^{-1}}$ preserves the integral, we get

$$\int \sigma_{h^{-1}}(a_h)a_{h^{-1}}\,d\mu.$$

So these are the same. $\square$

Now we can define $L^2(\Gamma \circlearrowright X)$, the completion of $(A\Gamma, \langle\cdot,\cdot\rangle_\tau)$. This completion is naturally isomorphic to $\ell^2(\Gamma, L^2(X))$. Alternatively, we can identifying it with the following: $\bigoplus_{g\in\Gamma}(L^2(X,\mu))_g$. We can also write it like $\{\sum_g \xi_g u_g : \xi_g \in L^2(X), \int\sum_{g\in\Gamma}|\xi_g|^2\,d\mu < \infty\}$. That is, we want $\sum_{g\in\Gamma}\|\xi_g\|^2_{L^2(X)} < \infty$.

Example 24.1. If X is a single point, then $A = \mathbb{C}$. So this gives $\ell^2(\Gamma)$.

On $L^2(\Gamma \circlearrowright X)$, we define operators of left multiplication and right multiplication by elements in $A\Gamma$; this gives $x = \sum_g c_g u_g \mapsto \lambda(x)$ or $\rho(x)$. In particular, the operation is

$$\lambda(a_g u_g)\sum_h \xi_h u_h = \sum_h a_g u_g \xi_h u_h = \sum_h a_g\sigma_g(\xi_h)u_{gh} = \sum_{h'} a_g\sigma_g(a_{g^{-1}h'})u_{h'}.$$

So we have two representations of the *-algebra $A\Gamma$ on $L^2(\Gamma \circlearrowright X)$. Check that $\lambda(x^*) = \lambda(x)^*$ and $\tau(x) = \langle\lambda(x)\hat{1},\hat{1}\rangle_\tau$, where $\hat{1}$ is the series with the only nonzero coefficient u_1 and ξ_1 to be the constant 1 function.

Definition 24.2. The **group measure space construction** is the von Neumann algebra $L(\Gamma \circlearrowright X) := \overline{\lambda(A\Gamma)}^{\mathrm{wo}} \subseteq \mathcal{B}(L^2(\Gamma \circlearrowright X))$ (and similarly for R).

24.3 Properties of the algebra

We can extend τ to the whole space by $\tau(x) := \langle x(\hat{1}),\hat{1}\rangle$.

Theorem 24.2. *$(L(\Gamma \circlearrowright X),\tau)$ and $(R(\Gamma \circlearrowright X),\tau)$ are tracial von Neumann algebras (and thus finite) with the faithful, normal trace τ.*

Theorem 24.3. *The left and right group measure space constructions are each other's commutants. That is, $L(\Gamma \circlearrowright X)' = R(\Gamma \circlearrowright X)$ and $R(\Gamma \circlearrowright X)' = L(\Gamma \circlearrowright X)$.*

In fact, if $\xi = \sum_g \xi_g u_g \in L^2(\Gamma \circlearrowright X)$ and η is similar, then we have the formal product $\xi\eta \in \ell^\infty(\Gamma, L^1(X))$. Then we call ξ a convolver if $\xi\eta \in L^2(\Gamma \circlearrowright X)$ for all $\eta \in L^2(\Gamma \circlearrowright X)$. Then we get a characterization of $L(\Gamma \circlearrowright X)$ in terms of left multiplication by convolvers, just like in the $L(\Gamma)$ case.

Observe that $A = L^\infty(X, \mu)$ sits inside $L(\Gamma \circlearrowright X)$ as the algebra $a \mapsto au_e$.

Theorem 24.4. *Let $\Gamma \circlearrowright X$ be a measure-preserving action.*

1. *$A \subseteq M = L(\Gamma \circlearrowright X)$ is **maximal abelian** in M (i.e. $A' \cap M = A$) if and only if $\Gamma \circlearrowright X$ is **essentially free** (i.e. $\mu(\{t \in X : gt = t\}) = 0$ for all $g \neq e$).*

2. *If $\Gamma \circlearrowright X$ is essentially free, then M is a factor if and only if σ is **ergodic** (if $a \in L^\infty(X)$ and $\sigma_g(a) = a$ for all g, then $a \in \mathbb{C}1$). So if Γ is infinite, then M is a II_1 factor.*

25 Student Presentations

In this class, every enrolled student gave a presentation on a topic. Here are notes I took for each presentation.

25.1 Kadison's transitivity theorem

Definition 25.1. If M is a C^*-algebra acting on a Hilbert space H, M is said to act **topologically irreducibly** if H has no proper, closed, invariant subspaces under M. M is said to act **algebraically irreducibly** if H has no proper, invariant subspaces under M.

From the definitions, we have that algebraically irreducible C^*-algebras are topologically irreducible.

Theorem 25.1 (Kadison's transitivity theorem)**.** *If M is topologically irreudicble, it is algebraically irreducible.*

Why is this called the transitivity theorem? We will show that M acts n-transitively on H; i.e. for all linearly independent $x_1, \ldots, x_n \in H$ and any $y_1, \ldots, y_n \in H$, there is an $A \in M$ such that $Ax_i = y_i$ for all $1 \leq i \leq n$.

Lemma 25.1. *Let $x_1, \ldots, x_n \in H$ be orthonormal, and let $z_1, \ldots, z_n \in H$ with $\|z_i\| \leq r$. Then there exists an operator $B \in \mathcal{B}(H)$ such that $Bx_i = z_i$ for all i and $\|B\| \leq \sqrt{2n}r$. If there is a selfadjoint T with $Tx_i = z_i$, then we can take B to be self-adjoint.*

Proof. Extend $x_1, \ldots, x_n, x_{n+1}, \ldots, x_m$ to an orthonormal basis for $\mathbb{C}\{x_1, \ldots, x_n, z_1, \ldots, z_n\}$ $(m < 2n)$. Let $\widetilde{B}$ be the matrix induced by splitting up the z_i according to this basis. Then

$$[\widetilde{B}] = \sqrt{\sum |\alpha_{i,j}|^2} \leq (2n \cdot r^2)^{1/2} = \sqrt{2n}r.$$

Extend it by making it 0 on the orthogonal complement. □

Proof. Assume $x_1, \ldots, x_n$ are orthonormal, so $x_1, \ldots, x_n \xrightarrow{B} y_1, \ldots, y_n$. By changing basis and conjugating by change of basis operators, we can get this result for arbitrary sets. Choose B_0 such that $B_0 x_i = y_i$. Take $A_0 \in M$ such that $\|A_0 x_i = y_i\| \leq \frac{1}{2\sqrt{2n}}$; this is possible because M is topologically irreducible. Choose B_1 such that $B_1 x_i = y_i - A_0 x_i$ and $\|B_1\| \leq \frac{1}{2}$. By Kaplansky's density theorem, choose $A_1 \in M$ such that $\|A_1\| \leq \frac{1}{2}$ and $A_1 x_i - B_1 x_i\| \leq \frac{1}{4\sqrt{2n}}$.

Continue recursively: Suppose we have defined B_k such that $\|B_k\| \leq \frac{1}{2^k}$ and $B_k x_i = y_i - A_0 x_i - A_1 x_i - \cdots - A_{k-1} x_i$. Choose $A_k \in M$ such that $\|A_k\| \leq \frac{1}{2^k}$, $\|A_k x_i - B_k x_i\| \leq \frac{1}{2^{k+1}\sqrt{2n}}$. Choose $\|B_{k+1}\| \leq \frac{1}{2^{k+1}}$ with $B_{k+1} x_i = y_i - A_0 x_i - A_1 x_i - \cdots - A_k x_i$. If $Tx_i = y_i$, we can choose the B_k and thus the A_k to be self-adjoint by Kaplansky's theorem. Let $A = \sum_{k=0}^{\infty} A_k$, This converges in norm to an element of M. Moreover,

$$y_i - Ax_i = y_i - \sum_{k=0}^{\infty} A_k x_i = \lim_k (y_i - a_0 x_i - A_1 x_i - \cdots - A_k x_i) = \lim_k (B_{k+1} x_i) = 0$$

because $\|x_i\| = 1$ and $\|B_{k+1}\| \leq 1/2^{k+1}$. This proves n-transitivity and thus Kadison's theorem. □

25.2 Dixmier's averaging theorem

Theorem 25.2 (Dixmier's averaging theorem)**.** *Let M be a von Neumann algebra with center $Z(M)$. For each $x \in M$, denote by $\overline{K(x)}$ the norm closure of the convex hull of $\{uxu^* : u \in U(M)\}$. Then $\overline{K(x)} \cap Z(M) \neq \varnothing$.*

The bulk of the proof is in the following lemma.

Lemma 25.2. *If $x = x^* \in M$, there is a $u \in U(M)$ and $y = y^* \in Z(M)$ such that*

$$\left\| \frac{1}{2}(x + u^* x u) - y \right\| \leq \frac{3}{4}\|x\|.$$

Proof. Suppose $\|x\| = 1$. Define projections $p = \mathbb{1}_{[0,1]}(x)$ and $q = \mathbb{1}_{[-1,0]}(x)$. By the comparison theorem, there exists some $z \in P(Z(M))$ such that $zq \prec zp$ and $(1-z)p \prec (1-z)q$. Take p_1, p_2, q_1, q_2 such that $zq \sim p_1 \leq p_1 + p_2 = 2p$ and $(1 - zp) \sim q - 1 \leq q_1 + q_2 = (1-z)q$.

Take two partial isometries $v, w \in M$ with $c^* c = w$ and $vv^* = p$, $w^* w = (1-z)p$, $vv^* = q$. Define $u = v + v^* + w + w^* + p_2 + q_2$. Then

$$\begin{aligned} u &= v^* v + vv^* + w^8 w + ww^* + q_2 + p_2 \\ &= zq + p_2 + (1-z)p + q_1 + q_2 + p_2 \\ &= p + q \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} u^* p_1 u &= zq, \qquad & u^* q_1 u &= (1-z)p \qquad & u^* p_2 u &= p_2, \\ u^* zqu &= p_1, \qquad & u^*(1-z)pu &= q_1, \qquad & u^* q_2 u &= q_2. \end{aligned}$$

We have $-zq \leq zx \leq zp = p_1 + p_2$. So

$$\begin{aligned} &\implies -p_1 \leq zu^* xu \leq zq + p_2 \\ &\implies -\frac{1}{2}(zq + p_1) \leq \frac{1}{2}(zx + zu^* xu) \leq \frac{1}{2} zq + p_1 + p_1 \end{aligned}$$

$$\begin{aligned}
&\implies \frac{1}{2}z \le \frac{1}{2}(2x + zu^*xu) \le z \\
&\implies -\frac{3}{4} \le \frac{1}{2}(2x - zu^*xu) - \frac{1}{4}z \le \frac{3}{4}z.
\end{aligned}$$

Similarly, repeating this with $1-z$ gives

$$-\frac{3}{4}(1-z) \le \frac{1}{2}((1-z)x + (1-z)u^*xu) + \frac{1}{4}(1-z) \le \frac{3}{4}(1-z).$$

If we add these together, we get

$$\left\| \frac{1}{2}(z + u^*xu) - \frac{2z-1}{4} \right\| \le \frac{3}{4}. \qquad \square$$

Proof. Let K denote the set of maps $\alpha : M \to M$ of the form $\alpha(x) = \sum_{i=1}^n c_i u_i^* x u_i$ with $u_i \in U(M)$, $\sum_i c_i = 1$ and $c_i \ge 0$. For general $x \in M$ denote $a_0 = \mathrm{Re}(x)$ and $b_0 = \mathrm{Im}(z)$. By the lemma, there exist some $u \in U(M)$ and $y_1 = y_1^* \in Z(M)$ with

$$\left\| \frac{1}{2}(a_0 + u^*a_0u) - y_1 \right\| \le \frac{3}{4}\|a_0\|.$$

Denote $\alpha_1(x) = \frac{1}{2}(x + u^*xu)$ and $a_1 = \alpha_1(a_0)$. Use the lemma again on $a_1 - y_1$. Continue inductively.

Given any $\varepsilon > 0$, we can find $\alpha \in K$ and $y \in Z(M)$ for which $\|\alpha(a_0) - y\| < \varepsilon$. Similarly, given this α, we can find $\beta \in K$ and $z \in Z(M)$ for which $\|\beta(\alpha(b_0)) - z\| < \varepsilon$. Thus,

$$\|\beta(\alpha(a_0)) - y\| = \|\beta(\alpha(a_0) - y)\| \le \|\alpha(a_0) - y\| < \varepsilon.$$

Therefore,

$$\|\beta(\alpha(x)) - (y + iz)\| < 2\varepsilon$$

The problem is that $y + iz$ might be dependent on ε. To fix that, we define a sequence $(\Gamma_n) \subseteq K$ and $(z_n) \subseteq Z(M)$ such that if $x_0 = x$ and $x_n = \gamma_n(x_{n-1})$, we have $\|x_n - z_n\| \le \frac{1}{2^n}$. Thus,

$$\|x_{n+1} - x_n\| = \|\gamma_{n+1}(x_n - z_n) - (x_n - z_n)\| \le \|\gamma_{n+1}(x_n - z_n)\| + \|x_n - z_n\| < \frac{1}{2^{n-1}}.$$

Thus, $x_n \to x$ and $z_n \to x$, so $x \in \overline{K(x)} \cap Z(M)$. $\square$

25.3 The Ryll-Nardzewski fixed point theorem

I gave this presentation. See my notes on the subject.

25.4 $\ell^1(\mathbb{Z})$ is not a C^*-algebra

Theorem 25.3. *$\ell^1(\mathbb{Z})$ is not a C^*-algebra.*

Theorem 25.4. *Let $\varphi \in C(S^1)$ with $\varphi(z) = 0$ for all $z \in S^1$. Then $\widehat{\varphi} \in \ell^\infty(\mathbb{Z})$, $\widehat{x\varphi} \in \ell^\infty(\mathbb{Z})$.*

These are consequences of the following fact.

Theorem 25.5. *Let $\Omega(\ell^1(\mathbb{Z}))$ be the maximal ideal space of $\ell^1(\mathbb{Z})$. $\Omega(\ell^1(\mathbb{Z})) \cong S^1$, where $\Omega(\ell^1(\mathbb{Z}))$ is equipped with the weak topology in $(\ell^1(\mathbb{Z}))^* \cong \ell^\infty(\mathbb{Z})$.*

Proof. Let i denote the natural isomorphism from $(\ell^1(\mathbb{Z}))^* \to \ell^\infty$. We claim that $i(\Omega(\ell^1(\mathbb{Z}))) = \{\alpha \in \ell^\infty(\mathbb{Z}) : \alpha(m+n) = \alpha(m)\alpha(n)\}$.

For any $\varphi \in \Omega(\ell^1(\mathbb{Z}))$ with $i(\varphi) = \alpha\varphi$,

$$\alpha\varphi(m+n) = \sum \delta_{m+n}\alpha_\varphi = \varphi(\delta_{m+n}) = \varphi(\delta_m * \delta_n) = \varphi(\delta_m)\varphi(\delta_n) = \alpha_\varphi(m) \cdot \alpha_\varphi(n).$$

On the other hand, if $\alpha(m+n) = \alpha(m) \cdot \alpha(n)$, then

$$\begin{aligned}
i^{-1}(\alpha)(f * g) &= \sum (f * g)\alpha \sum_i \sum_j f(i-j)g(j)\alpha(i) \\
&= \sum_j \sum_i f(i-j)g(j)\alpha(i-j)\alpha(j) \\
&= \langle g, \alpha \rangle \langle f, \alpha \rangle \\
&= i^{-1}(\alpha(f)) \cdot i^{-1}(\alpha(g)).
\end{aligned}$$

Now observe that $\alpha(m) = (\alpha(1))^m$, which gives a bijection $\widehat{\mathbb{Z}} \to A^1$ by $\alpha \mapsto \alpha(1)$. These spaces are compact, so we only need to check continuity of the map to get a homeomorphism. If $\alpha_i \xrightarrow{wk} \alpha$, then

$$\alpha_i(1) = \sum \delta_1 \alpha_i \to \sum \delta_1 \alpha = \alpha(1).$$

So we get that $S^1 \cong \widehat{\mathbb{Z}} \cong \Omega(\ell^1(\mathbb{Z}))$. $\square$

Now we can show that $\ell^1(\mathbb{Z})$ is not a C^*-algebra.

Proof. Assume $\ell^1(\mathbb{Z})$ is a C^*-algebra. Then by the Gelfand transform, $\ell^1(\mathbb{Z}) \cong C(S^1)$. Then $\Gamma(\ell^1(\mathbb{Z})) = \{\varphi \in C(S^1) : \widehat{\varphi} \in \ell^1(\mathbb{Z})\}$.

We claim that $\widehat{\Gamma}(f) = f$, where $f \in \ell^1(\mathbb{Z})$. If $\Gamma(f) \in C^1(S^1)$, then $\Gamma(f)(z) = \langle f, z^n \rangle = \sum f(n)z^n$. We check

$$\widehat{\Gamma(f)}(n) = \frac{1}{2\pi} \int_0^{2\pi} \sum f(n)e^{inx})e^{inx}\, dx = f(n).$$

We now claim that if $\varphi \in C(S^1)$ then $\widehat{\varphi} \in \ell^1(\mathbb{Z})$. We have

$$^\wedge(\Gamma(\widehat{\varphi}) - \varphi) - \widehat{\Gamma(\widehat{\varphi})} - \widehat{\varphi} = 0$$

by the first claim. □

Here is the proof of the other result.

Proof. $\Gamma(f)$ is invertible if and only if f is invertible. Then if $\varphi = \Gamma(f)$, then $1/\varphi = \Gamma(f^{-1})$. □

25.5 There are no nontrivial projections in $C^*_{\text{red}}(\mathbb{F}_2)$.

This is a presentation about Effros' paper, "Why the circle is connected." In a more concrete sense, this is about the fact that the reduced C^*-algebra of $\mathbb{F}_n$ has no nontrivial projections.

To begin, let's motivate and define a connected C^*-algebra by examining $C(X)$ for a compact topological space X. If X is connected and $P \in \operatorname{Proj}(C(X))$, then $P = 0$ or 1. This is because $P^{-1}((1/2, \infty))$ and $P^{-1}((-\infty, 1/2))$ cover X. So we define connectedness for a C^*-algebra as follows.

Definition 25.2. A C^*-algebra M is **connected** if $\operatorname{Proj}(M) = \{I, 0\}$.

Consider $C(S^1)$. By Fourier series, $C(S^1) \cong C^*_{\text{red}}(\mathbb{Z})$, the reduced C^*-algebra. $\mathbb{Z}$ is the free group on 1 generator. This is why this is related to the circle.

Theorem 25.6. *There are no nontrivial projections in* $C^*_{\text{red}}(\mathbb{F}_2)$.

With slight modifications, the argument we will make will generalize, with modifications to $\mathbb{F}_n$. The idea is that we will get two traces on $C^*_{\text{red}}(\mathbb{F}_2)$, one of which (ordinary trace - tr) is always $\mathbb{Z}$-valued on projections, and the other, τ, is faithful and unital. Then we will find τ in terms of tr and use the following lemmas.

Lemma 25.3. *If* τ *is*

1. *faithful* ($\tau(a^*a) \geq 0$ *for all* a *with* $\tau(a^*a) = 0 \implies a = 0$*),*
2. *unital* ($\tau(1) = 1$*),*
3. *tracial* ($\tau(ab) = \tau(ba)$*),*

then $\tau(\operatorname{Proj}(M)) \subseteq \mathbb{Z}$*. So there are no nontrivial projections.*

Proof. Let P be a projection. Since $P^*P = P$, $\tau(P^*P) = \tau(P) \geq 0$ (with equality iff $P = 0$). The same is true for $1 - P$. So $0 \leq P \leq 1$. But $\tau(P) = 0$ or 1. If the former occurs, then since τ is faithful, $P = 0$. If the latter occurs, $\tau(1 - P(= \tau(1) - \tau(P) = 0$, so $1 - P = 0$. So $P = 1$. □

Lemma 25.4. *Let P, Q be projections in $\mathcal{B}(H)$ and suppose that $P - Q$ is trace class (i.e. $\operatorname{tr}(|P-Q|) < \infty$, so $\operatorname{tr}(P-Q)$ is independent of basis). Then $\operatorname{tr}(P-Q) \in \mathbb{Z}$.*

Here, $\operatorname{tr}(A) = \sum_k \langle e_k, Ae_k \rangle$, where e_k is an orthonormal basis of the space.

Proof. First, note that

$$\begin{aligned} P(P-Q)^2 &= P(P+Q-PQ-QP) \\ &= P + PQ - PQ - PQP \\ &= P - PQP, \end{aligned}$$

$$\begin{aligned} (P-Q)^2 P &= (P+Q-PQ-QP)P \\ &= P - PQP. \end{aligned}$$

So $P(P-Q)^2 = (P-Q)^2 P$. Similarly, $Q(P-Q)^2 = (P-Q)^2 Q$. So $(P-Q)^2$ is positive, and $\operatorname{tr}((P-Q)) < \infty$, so $(P-Q)^2$ is a Hilbert-Schmidt operator. So $(P-Q)^2$ is compact.

We get from the spectral theorem that

$$(P-Q)^2 = \sum_k \lambda_k P_k,$$

where λ_k are eigenvalues with $\lambda_k > 0$ and P_k are projections. We can take $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$, and we have $\lim_{k\to\infty} \lambda_k = 0$. Define $q = 1 - \sum_k P_k$. Observe that $(P-Q)q = 0$, as

$$\begin{aligned} \langle P-Q)qx, (P-Q)qx \rangle &= \langle qx, (P-Q)^2 qx \rangle \\ &= \left\langle qx, \sum_k \lambda_k P_k qx \right\rangle \\ &= 0, \end{aligned}$$

where $P_k q = 0$ for all k. Also, $PP_k = P_k P$ and $QP_k = P_k Q$ (as $P_k = f_k((P-Q)^2)$ for some continuous f_k on $\operatorname{Spec}((P-Q)^2)$). So

$$P - Q = \sum_k (P-Q)P_k = \sum_k PP_k - \sum_j QP_j.$$

This gives us that

$$\operatorname{tr}(P-Q) = \sum \operatorname{tr}((P-Q)P_k) = \sum_k \operatorname{tr}(PP_k) - \operatorname{tr}(QP_k).$$

Moreover, $\operatorname{tr}(PP_k)$ and $\operatorname{tr}(QP_k)$ are integers because they are finite dimensional projections (The trace of a finite dimensional projection is its dimension.) $\square$

We can show that a set is connected by showing that any probability measure on the set gives 0 or 1 measure to a clopen set. This will be similar to what we are doing. The trace we will define will be analogous to integrating against Lebesgue measure.

The second trace we are interested in is $\tau_{\mathbb{F}_2}$ given by

$$\tau(a) = \langle e_1, \lambda(a)e_1\rangle$$

where $a \in C^*_{\mathrm{red}}(\mathbb{F}_2)$, and λ is left multiplication.

Remark 25.1. Suppose $a = \sum a_g \ell_g$. Then $\langle e_h, a_g e_{gh}\rangle = a_g \delta_{h,gh}$, so $g = 1 = a_1$. Such a trace is faithful and unital (done in class).

Compare this to $\mathrm{tr}(a)$: The trace tr is a sum of terms like

$$\langle e_g, \lambda(a)e_g\rangle = \langle e_1, \lambda(a)e_1\rangle = \tau(a).$$

We have $\lambda : C^*_{\mathrm{red}}(\mathbb{F}_2) \to \mathcal{B}(\ell^2(\mathbb{F}_2))$. We will define $\lambda_0 : C^*_{\mathrm{red}}(\mathbb{F}_2) \to \mathcal{B}(\ell^2(\mathbb{F}_2))$ as follows. Write $\mathbb{F}_2 = S_u \cup S_v \cup \{e\}$, where S_u is the set of words that end with u or u^{-1} and S_v is the set of words that end with v or v^{-1}. This gives $\ell^2(\mathbb{F}_2) = H_u \oplus H_V \oplus \mathbb{C}e_1$.

H_u and H_v are isomorphic to $\mathbb{F}_2$ in the sense that $\mathbb{F}_2$ acts in the same way on them. What is the action/representation? Define

$$\lambda_0(u)e_1 = 0, \qquad \lambda_0(u)e_g = \begin{cases} e_{ug} & \text{if } g \neq u^{-1} \text{ or } 1 \\ e_u & g = u^{-1} \end{cases}, \qquad \lambda_0(v)e_g = \begin{cases} e_{vg} & \text{if } g \neq v^{-1} \text{ or } 1 \\ e_u & g = u^{-1}. \end{cases}$$

This defines a representation $\lambda_0 : C^*_{\mathrm{red}}(\mathbb{F}_2) \to \mathcal{B}(\ell^2(\mathbb{F}_2))$ where $\lambda_0(e_1) = 0$; this is the only thing in the kernel. We have $\lambda|_{H_u} = \lambda_0|_{H_u}$ and $\lambda|_{H_v} = \lambda_0|_{H_v}$. So we get

$$\lambda_0 \cong \lambda|_{H_u} \oplus \lambda|_{H_v} \oplus 0_{e_1}.$$

Let's compare λ and λ_0. If $a \in C^*_{\mathrm{red}}(\mathbb{F}_2)$, then $\tau(a) = \langle e_g, \lambda(a)e_g\rangle$. On the other hand, if $s \in S_u$, then

$$\langle e_s, \lambda_0(u)e_s\rangle = \langle e_t, \lambda(u)e_t\rangle = \tau(u)$$

for some t. But $\langle e_1, \lambda_0(u)e_1\rangle = 0$. So the diagonal entries of $\lambda(u)$ and $\lambda_0(u)$ differ only at e_1. By induction on the length of a word, we get that $\lambda(a)$ and $\lambda_0(a)$ differ only at finitely many places for $a \in \mathbb{C}\mathbb{F}_2$.

We want to say that $\tau(a) = \mathrm{tr}(\lambda(a) - \lambda_0(a))$. But we don't know that $\lambda(a) - \lambda_0(a)$ is trace class in general. Define $\mathscr{A}_0 \subseteq C^*_{\mathrm{red}}(\mathbb{F}_2)$ by $a \in A_0$ if and only if $\lambda(a) - \lambda_0(a)$ is trace class. We want to show that if $P \in \mathrm{Proj}(C^*_{\mathrm{red}}(\mathbb{F}_2))$ with $P \neq 0, I$, then there exists an $e \in \mathrm{Proj}(\mathscr{A}_0)$ with $e \neq 0, I$. But then $\tau(e) = \mathrm{tr}(\lambda(e) - \lambda_0(e)) \in \mathbb{Z}$, which is a contradiction by the first lemma.

If $a \in \mathbb{C}\mathbb{F}_2$, $\lambda(a)$ and $\lambda_0(a)$ differ at only finitely many places, so $\lambda(a) - \lambda_0(a)$ is finite rank and is hence trace class. So $\mathbb{C}\mathbb{F}_2 \subseteq \mathscr{A}_0$. Since $\mathbb{C}\mathbb{F}_2$ is dense in $C^*_{\mathrm{red}}(\mathbb{F}_2)$, we have an

$a \in \mathbb{C}\mathbb{F}_2$ s.t. $\|a - P\| < 1/3$; we can choose a to be self-adjoint. Then $\mathrm{Spec}(a)$ is contained in the union of some neighborhood in $\mathbb{R}$ about 0 and some neighborhood in $\mathbb{R}$ about 1.

We can't just use continuous functional calculus, because we might get something out of it which is not trace class. Instead, define e by

$$e = \frac{1}{2\pi i} \int_\Gamma (z - a)^{-1}\, dz,$$

where Γ is a closed contour around the part of $\mathrm{Spec}(a)$ near 1. Since $\lambda(a)$ and $\lambda_0(a)$ differ at only finitely many places, the same is true for $\lambda(z - a)$ and $\lambda_0(z - a)$ (and we have a uniform bound on the dimension of $\ker(\lambda - \lambda_0)^\perp$). Let R_n be the n-th approximating Riemann sum. Then $R_n \to e$ in the operator norm topology, so $|R_n| \to |e|$ in the operator norm. Also, $\|R_n\| \leq C$ for all n. Moreover, if A is finite rank, $\mathrm{tr}(|A|) \leq \|A\| \cdot \dim(\mathrm{im}(A))$, which gives us a uniform bound on $\mathrm{tr}(|\lambda((z - a)^{-1}) - \lambda_0((z - a)^{-1})|)$. Then use the fact that if $A_n \to A$ in the operator norm and $\mathrm{tr}(|A|) \leq C$, then $\mathrm{tr}(|A|) \leq C$. Hence, e is trace class, so $\tau(e) = \mathrm{tr}(\lambda(e) - \lambda_0(e) \in \mathbb{Z}$.